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A FAMILY OF M-SURFACES WHOSE AUTOMORPHISM GROUPS ACT TRANSITIVELY ON THE MIRRORS

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Abstract

Let X be a compact Riemann surface of genus $g > 1$. A *symmetry* T of X is an anticonformal involution. The fixed point set of T is a disjoint union of simple closed curves, each of which is called a *mirror* of T. If T fixes $g+1$ mirrors then it is called an *M-symmetry* and X is called an *M-surface.* If X admits an automorphism of order $g+1$ which cyclically permutes the mirrors of T then we shall call X an M-surface with the *M-property.* In this paper we investigate those M-surfaces with the M-property and their automorphism groups.

1 Introduction

Let X be a compact Riemann surface of genus $g > 1$. X is called *symmetric* if it admits an anticonformal involution $T: X \to X$ which we call a *symmetry* of X. The fixed point set of T consists of k simple closed curves, each of which is called a *mirror* of T. Here k is a positive integer and by the Harnack's theorem $0 \leq k \leq g+1$. If T has $g+1$ mirrors then it is called an *M-symmetry* and X is called an *M-surface.* If X admits an automorphism of order $g+1$ which cyclically permutes the mirrors of T then we will say that X is an M-surface with the *M-property.* In section 2 we give the background material. Our aim in this paper is to investigate those M-surfaces with the M-property and their automorphism groups, which is discussed in section 3. We consider hyperelliptic and non-hyperelliptic M-surfaces in different cases and give our main results in Theorem 3.3 and Theorem 3.5 which we state below:

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Theorem 3.3. *Let* $X = \mathcal{U}/K$ *be a hyperelliptic M-surface of genus* $g > 1$ *with the M-property. Then* K *is always contained as a normal subgroup of index* $8g + 8$ *in an NEC group* Δ*, where* Δ *has signature* $(0, +, [-], \{(2^{(3)}, g+1)\})$ *and* Δ/K *is isomorphic to* $C_2 \times C_2 \times D_{g+1}$ *and contained in* $Aut^{\pm}(X)$.

Theorem 3.5. *Let* X *be a non-hyperelliptic M-surface of genus* $g > 1$ *(*g odd*) with the M-property and* $T : X \to X$ *be the M-symmetry. If* $X/ < T >= U/\Gamma$ *then* Γ *is always contained as a normal subgroup of index* $2g + 2$ *in an NEC group* Δ*, where* Δ *has signature* $(0, +, [-], \{(2^{(4)}, (g+1)/2)\})$ *and* Δ/Γ *is isomorphic to* D_{g+1} *and contained in* $Aut(X/ < T >)$.

2 Preliminaries

Non-Euclidean Crystallographic Groups. Let $\mathcal{U}$ denote the upper-half complex plane and $\mathcal{L}$ denote the group of conformal and anticonformal homeomorphisms of $\mathcal{U}$. A *non-Euclidean crystallographic* (NEC) group is a discrete subgroup Γ of $\mathcal{L}$ and we shall assume that $\mathcal{U}/\Gamma$ is compact. Let $\mathcal{L}^+$ be the subgroup of $\mathcal{L}$ consisting of conformal homeomorphisms. An NEC group contained in $\mathcal{L}^+$ is called a *Fuchsian group*, otherwise it is called a *proper NEC group*. The *signature* of an NEC group is defined to be

$$(g; \pm; [m_1, m_2, \dots, m_r]; \{(n_{11}, \dots, n_{1s_1}), \dots, (n_{k1}, \dots, n_{ks_k})\}). \quad (2.1)$$

The algebraic and geometric structure of an NEC group is completely determined by its signature. If Γ has signature (2.1) then $\mathcal{U}/\Gamma$ is a compact surface of genus g with k holes. The surface is orientable if $+$ sign is used and non-orientable if $-$ sign is used. The integers $m_1, m_2, \dots, m_r$ are called the *proper periods* and represent the branching over interior points of $\mathcal{U}/\Gamma$ in the natural projection from $\mathcal{U}$ to $\mathcal{U}/\Gamma$. The brackets $(n_{i1}, \dots, n_{is_i})$ are called the *period cycles* and the integers $n_{i1}, \dots, n_{is_i}$ are called *link periods* and they represent the branching around the ith hole. The subgroup Γ^+ of Γ consisting of orientation preserving transformations is called the *canonical Fuchsian group* of Γ. Now let us describe the presentation of a group with signature (2.1). If the $+$ sign is used, it has canonical generators

(i) $x_1, \ldots, x_r$ (elliptic elements),

(ii) $c_{10}, \ldots, c_{1s_1}, \ldots, c_{k0}, \ldots, c_{ks_k}$ (reflections),

(iii) $e_1, \ldots, e_k$ (usually hyperbolic elements but sometimes elliptic),

(iv) $a_1, b_1, \ldots, a_g, b_g$ (hyperbolic elements),

and relations

(a) $x_i^{m_i} = 1$, for $i = 1, \ldots, r$,

(b) $c_{i,j-1}^2 = c_{ij}^2 = (c_{i,j-1}c_{ij})^{n_{ij}} = 1$, for $i = 1, \ldots, k$ and $j = 1, \ldots, s_i$, we shall call $c_{i,j-1}c_{ij}$ linked reflection generators with link period n_{ij}.

(c) $e_i^{-1}c_{i0}e_i = c_{is_i}$ for $i = 1, \ldots, k$,

(d) $x_1x_2 \ldots x_r e_1 e_2 \ldots e_k a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_g b_g a_g^{-1} b_g^{-1} = 1$.

If there is $-$ sign in the signature we replace (iv) by

(iv)′ $a_1, , \ldots, a_g$ (glide reflections), and (d) by

(d)′ $x_1x_2 \ldots x_r e_1 e_2 \ldots e_k a_1^2 a_2^2 \ldots a_g^2 = 1$. See [5], [7] and [12] for details.

If Γ is an NEC group with signature (2.1) then the non-Euclidean area of a fundamental region for Γ is given by

$$\mu(\Gamma) = 2\pi\left(\alpha g - 2 + k + \sum_{i=1}^{r}(1 - \frac{1}{m_i}) + \sum_{i=1}^{k}\sum_{j=0}^{s_i}\frac{1}{2}(1 - \frac{1}{n_{ij}})\right)$$

where $\alpha = 2$ if the sign is $+$ and $\alpha = 1$ if the sign is $-$ in the signature of Γ. Since Γ is an NEC group then $\mu(\Gamma) > 0$, (see [12]).

A Fuchsian group of signature $(0; +; [l, m, n])$ (for short $[l, m, n]$)) is called a *triangle* group, where $1/l + 1/m + 1/n < 1$.

For convenience we make abbreviations such as

$$(5; +; [2^{(3)}, 3^{(2)}]; \{(4^{(3)}), (\ )^{(4)}\})$$

for

$$(5; +; [2, 2, 2, 3, 3]; \{(4, 4, 4), (\ ), (\ ), (\ ), (\ )\}).$$

Symmetries and Automorphisms of Riemann Surfaces. It is well-known that every Riemann surface X is conformally equivalent to the quotient of the upper-half complex plane $\mathcal{U}$ by a torsion free Fuchsian group K. In this paper we shall deal with compact Riemann surfaces and so K will contain no parabolic transformations. Such Fuchsian groups are called *surface groups.* An automorphism of X is a conformal or anticonformal homeomorphism $f: X \to X$. All automorphisms of X form a group under composition of maps and we shall denote it by $AutX$ and the subgroup consisting of conformal automorphisms by Aut^+X. A finite group G acts as a group of automorphisms of a Riemann surface $X = \mathcal{U}/K$ of genus $g > 1$ if and only if G is isomorphic to the factor group Γ/K, where Γ is an NEC group containing K as a normal subgroup. So we can find an epimorphism from Γ to G with kernel K. Such an epimorphism is called a *surface kernel epimorphism.* It is also known that Aut^+X and $AutX$ are isomorphic to $N^+(K)/K$ and $N(K)/K$ respectively, where $N^+(K)$ and $N(K)$ denote the normalisers of K in $\mathcal{L}^+$ and $\mathcal{L}$, respectively.

We now state the following theorem which is given in Natanzon [10] and [11], see also Bujalance and Costa [4].

Theorem 2.1. *Let X be a non-hyperelliptic M-surface. Then:*

(i) X admits exactly one M-symmetry,

(ii) $AutX = C_2 \times Aut^+X$, where C_2 is generated by the M-symmetry and Aut^+X, the group of conformal automorphisms of X, is isomorphic to a finite subgroup of the group of isometries of the sphere.

■

If Γ is an NEC group without elliptic elements then $\mathcal{U}/\Gamma$ is a *Klein surface.* By a Klein surface we mean a surface with a dianalytic structure [2]. It is known that every Klein surface S can be represented as $\mathcal{U}/\Gamma$ where Γ is an NEC group without elliptic elements. Γ may contain reflections, in such case S is a Klein surface with boundary. Any automorphism of S can be expressed as Δ/Γ where Δ is another NEC group containing Γ as a normal subgroup. The full group of automorphisms of S, $AutS$, is isomorphic to $N(\Gamma)/\Gamma$, where $N(\Gamma)$ denotes the normaliser of Γ in $\mathcal{L}$. Let Γ^+ be the subgroup of Γ consisting of orientation preserving elements. Then $S^+ = \mathcal{U}/\Gamma^+$ is a Riemann surface and known as the *complex double* of S [2]. S is isomorphic (dianalytically equivalent) to $S^+/\langle T\rangle$, where T is a symmetry of S^+. On the other hand, it is known that the automorphisms of S consist of conformal automorphisms of S^+ commuting with T [2, Theorem 1.11.1]. (For more details about Klein surfaces and their automorphisms see [2] and [5]).

In this paper the Hoare's theorem will be our main tool which gives us a procedure for calculating the signature of a subgroup Λ of a given NEC group Δ, knowing the action of the canonical generators of Δ on the Λ-cosets. For details see Hoare [6].

3 M-Surfaces with the M-property

Lemma 3.1. *Let Ω be a non-empty set and G be the group of all permutations of Ω. If $\alpha, \beta \in G$ and $\alpha\beta = \beta\alpha$, then α (respectively β) maps the fixed-point set of β (respectively α) to itself.*

■

Lemma 3.2. *Let X be a Riemann surface and $T: X \to X$ be a symmetry with $G = AutX$. If $\mathcal{M} = \{m_1, m_2, \ldots, m_k\}$ is the set of mirrors of T and $\mathcal{H} = \{g \in G \mid g(\mathcal{M}) = \mathcal{M}\}$, then $\mathcal{H} = C_G(T)$, the centraliser of T in G.*

Proof. Let $V \in C_G(T)$ then $TV = VT$ and by Lemma 3.1, V maps the fixed point set of T to itself and so $V \in \mathcal{H}$.

Now let $V \in \mathcal{H}$ and $m_i \in \mathcal{M}$ then $V(m_i) \in \mathcal{M}$ and $T(m_i) = m_i$.

$$
\begin{aligned}
lllV^{-1}TV(m_i) &= V^{-1}T(V(m_i)) \\
&= V^{-1}(V(m_i)) \qquad (V(m_i) \in \mathcal{M}) \\
&= m_i \\
&= T(m_i)
\end{aligned}
$$

So $TV^{-1}TV$ fixes m_i pointwise. As $TV^{-1}TV$ is conformal, $TV^{-1}TV = I$. Therefore, $V^{-1}TV = T$ and $V \in C_G(T)$.

■

So if X is an M-surface with the M-property and $T: X \to X$ an M-symmetry, then by Lemma 3.2 we get an induced action of $\mathcal{H}$ on $X/\langle T\rangle$ as follows, where $\mathcal{H}$ is the centraliser of T in $AutX$.

For every $g \in \mathcal{H}$,

$$g([x]_T) = [g(x)]_T \tag{3.1}$$

gives us an action of $\mathcal{H}$ on $X/\langle T\rangle$ where $[x]_T$ denotes a point on the surface$X/\langle T\rangle$. Since

$$g([Tx]_T) = [g(Tx)]_T = [Tg(x)]_T = [g(x)]_T = g([x]_T),$$

(3.1) is well-defined.

We know that $X/\langle T\rangle$ is a Klein surface of genus 0 with $g+1$ boundary components. Therefore, it can be uniformised by an NEC group, i.e. there is an NEC group Γ with signature

$$(0; +; [\ ]; \{(\ )^{(g+1)}\})$$

such that $X/\langle T\rangle$ is isomorphic (dianalytically equivalent) to $\mathcal{U}/\Gamma$.

As $\mathcal{H}$ acts on $X/\langle T\rangle$, there exists an NEC group Δ containing Γ as a normal subgroup of index $|\mathcal{H}|$ such that $\Delta/\Gamma \simeq \mathcal{H}$. Thus, there is an epimorphism $\theta: \Delta \to \mathcal{H}$ with kernel Γ. Now we want to find possible signatures for Δ. As X is an M-surface with the M-property, $\mathcal{H}$ has a cyclic subgroup H of order $g+1$ whose generators cyclically permute the mirrors of T. Then $\Lambda = \theta^{-1}(H)$ is an NEC group containing Γ with index $g+1$.

First, let us find the signature of Λ. Since the generators of H cyclically permute the boundary components of $X/\langle T\rangle$, the quotient surface $(X/\langle T\rangle)/\langle H\rangle$ will have at least one smooth boundary component. Therefore, the signature of Λ will contain at least one empty period cycle and possibly some non-empty period cycles. So the signature of Λ will be of the form

$$(h; \pm; [m_1, m_2, \ldots, m_n]; \{(\ )^k, (n_{11}, \ldots, n_{1s1}), \ldots, (n_{r1}, \ldots, n_{rsr})\}). \tag{3.2}$$

Since Γ is a normal subgroup of Λ with index $g+1$, by the Riemann-Hurwitz formula $\mu(\Gamma) = (g+1)\mu(\Lambda)$, we get

$$\frac{g-1}{g+1} = \delta h - 2 + k + r + \sum_{i=1}^{n}(1 - \frac{1}{m_i}) + \frac{1}{2}\sum_{j=1}^{r}\sum_{\mu=1}^{s_j}(1 - \frac{1}{n_{j\mu}}) \tag{3.3}$$

where δ is 2 if the sign in the signature is plus and 1 if the sign is minus. Since the left-hand side of (3.3) is less than 1, we have the following restrictions on h, k, r and n: $0 \leq h \leq 1$, $1 \leq k \leq 2$, $0 \leq r \leq 1$ and $n \leq 3$.

Under these restrictions, if we do calculations we shall find the following possible signatures for Λ:

$$\Lambda_1 = (0; +; [(g+1)^{(2)}]; \{(\ )\})$$
$$\Lambda_2 = (0; +; [\frac{g+1}{2}]; \{(\ )^{(2)}\})$$
$$\Lambda_3 = (1; -; [\frac{g+1}{2}]; \{(\ )\})$$
$$\Lambda_4 = (0; +; [\ ]; \{(\ ), ((\frac{g+1}{2})^{(2)})\}).$$

Now let us determine from which of these NEC groups there is an epimorphism to C_{g+1} such that the kernel has signature

$$(0; +; [\ ]; \{(\ )^{(g+1)})\}).$$

Let us begin with Λ_1.

Λ_1 and C_{g+1} have the following presentations:

$$\Lambda_1 : \langle x_1, x_2, c, e \mid x_1^{g+1} = x_2^{g+1} = c^2 = ece^{-1}c = x_1x_2e = 1\rangle$$
$$C_{g+1} : \langle \alpha \mid \alpha^{g+1} = 1\rangle$$

Let us define the epimorphism $\theta_1 : \Lambda_1 \to C_{g+1}$ as follows:

$$\theta_1 : \begin{cases} x_1 & \mapsto \alpha \\ x_2 & \mapsto \alpha^{-1} \\ c & \mapsto 1 \\ e & \mapsto 1. \end{cases}$$

Using the Hoare's theorem we can find that $Ker\theta_1$ has signature

$$(0; +; [\ ]; \{(\quad)^{(g+1)}\}).$$

Note that in the case above our aim was to find an epimorphism whose kernel is an NEC group with signature

$$(0; +; [\ ]; \{(\quad)^{(g+1)}\}).$$

So $\theta_1(c)$ must be the identity. Otherwise, there will be no reflection in the kernel of θ_1. To get $g+1$ chains, e also must map to the identity. x_1 can map to any element of order $g+1$ and x_2 to the inverse of that element. Therefore, θ_1 is the unique epimorphism (up to automorphism of C_{g+1}) whose kernel has signature

$$(0;+;[\,];\{(\quad)^{(g+1)}\})$$

in the sense that if $\theta_1^{'}:\Lambda_1 \to C_{g+1}$ is another such epimorphism, then there exists an automorphism f of C_{g+1} such that $\theta_1^{'} = f\theta_1$.

For example, if k and $g+1$ are coprime, then

$$\theta_1^{'}: \begin{cases} x_1 & \mapsto \alpha^k \\ x_2 & \mapsto \alpha^{-k} \\ c & \mapsto 1 \\ e & \mapsto 1 \end{cases}$$

is another epimorphism from Λ_1 to C_{g+1} whose kernel has signature

$$(0;+;[\,];\{(\quad)^{(g+1)}\}).$$

However, $f: C_{g+1} \to C_{g+1}$, $f(\alpha) = \alpha^k$ is an automorphism of C_{g+1} and $\theta_1^{'} = f\theta_1$.

We now do the same calculations to see whether there is an epimorphism from Λ_2 to C_{g+1} whose kernel has signature

$$(0;+;[\,];\{(\quad)^{(g+1)}\}).$$

Λ_2 and C_{g+1} have the following presentations:

$$\Lambda_2 : \langle x, c_1, c_2, e_1, e_2 \mid x^{\frac{g+1}{2}} = c_1^2 = c_2^2 = e_1c_1e_1^{-1}c_1 = e_2c_2e_2^{-1}c_2 = xe_1e_2 = 1\rangle$$

$$C_{g+1} : \ \langle \alpha \mid \alpha^{g+1} = 1\rangle$$

Let us define $\theta_2: \Lambda_2 \to C_{g+1}$ as follows:

$$\theta_2: \begin{cases} x & \mapsto \alpha^2 \\ c_1 & \mapsto 1 \\ e_1 & \mapsto 1 \\ c_2 & \mapsto \alpha^{\frac{g+1}{2}} \\ e_2 & \mapsto \alpha^{-2}. \end{cases}$$

Similarly, we can find that $Ker\theta_2$ has signature

$$(0;+;[\ ];\{(\ \ )^{(g+1)}\}).$$

Note that in this case unless $\frac{g+1}{2}$ is odd, 2 and $\frac{g+1}{2}$ are not coprime and hence α^2 and $\alpha^{\frac{g+1}{2}}$ cannot generate C_{g+1}. Also g must be odd. Otherwise $\frac{g+1}{2}$ will not be an integer. So if the above condition are not satisfied, then θ_2 cannot be an epimorphism. Similarly, we can show that θ_2 is the unique epimorphism (up to automorphism of C_{g+1}) from Λ_2 to C_{g+1} whose kernel has signature

$$(0;+;[\ ];\{(\ \ )^{(g+1)}\}).$$

We now search whether there is an epimorphism from Λ_3 to C_{g+1}. Λ_3 and C_{g+1} have the following presentations:

$$\Lambda_3: \ \langle a, x, c, e \mid x^{\frac{g+1}{2}} = c^2 = ece^{-1}c = xea^2 = 1\rangle$$

$$C_{g+1}: \ \langle \alpha \mid \alpha^{g+1} = 1\rangle$$

Let us define $\theta_3 \colon \Lambda_3 \to C_{g+1}$ as follows:

$$\theta_3: \begin{cases} x & \mapsto \alpha^2 \\ c & \mapsto 1 \\ e & \mapsto 1 \\ a & \mapsto \alpha^{-1}. \end{cases}$$

As before we can find that $Ker\theta_3$ has signature

$$(0;+;[\ ];\{(\ \ )^{(g+1)}\}).$$

In this case g must be odd. Otherwise $\frac{g+1}{2}$ will not be an integer.

Again, θ_3 is the unique epimorphism (up to automorphism of C_{g+1}) from Λ_3 to C_{g+1} whose kernel has signature

$$(0;+;[\ ];\{(\ \ )^{(g+1)}\}).$$

Lastly, we now show that there is no epimorphism from Λ_4 to C_{g+1} whose kernel has signature

$$(0;+;[\ ];\{(\ \ )^{(g+1)}\}).$$

Λ_4 and C_{g+1} have the following presentations:

$$\Lambda_4 : \langle c, e, c_0, c_1, c_2, e_1 \mid c^2 = c_0^2 = c_1^2 = c_2^2 = (c_0c_1)^{\frac{g+1}{2}} = (c_1c_2)^{\frac{g+1}{2}} = ece^{-1}c = e_1c_0e_1^{-1} = c_2 = ee_1 = 1\rangle$$

$$C_{g+1} : \langle \alpha \mid \alpha^{g+1} = 1\rangle$$

As in the previous case g must be odd and so the only element of order 2 in C_{g+1} is $\alpha^{\frac{g+1}{2}}$. As the kernel must contain reflections, one of the reflection generators of Λ_4 must map to the identity. So c_i $(i = 0, 1, 2)$ and c can map to either $\alpha^{\frac{g+1}{2}}$ or to the identity.

Assume that $\theta_4 \colon \Lambda_4 \to C_{g+1}$ is an epimorphism as required. Consider the reflection generators c_0 and c_1. Both of them cannot map to the same element for otherwise, there will be elliptic elements in the kernel, which is not allowed. Thus, one of them has to map to $\alpha^{\frac{g+1}{2}}$ and the other to the identity. However, in either case it follows from the relation $(c_0c_1)^{\frac{g+1}{2}} = 1$ that $\frac{g+1}{2} = 2$ and this is not true except for $g = 3$. In the case when $g = 3$, using the Hoare's theorem we can show that there is no epimorphism from Λ_4 to C_4 as required. Thus, for every $g > 1$ there is no epimorphism from Λ_4 to C_{g+1} whose kernel has signature

$$(0; +; [\ ]; \{(\ \ )^{(g+1)}\}).$$

As we shall see later the remaining three epimorphisms correspond to M-surfaces with the M-property. The epimorphism θ_1 corresponds to hyperelliptic M-surfaces of genus $g \geq 2$ while θ_2 and θ_3 correspond to non-hyperelliptic M-surfaces of odd genus with the M-property. Recall that surfaces corresponding to θ_2 must have genus $g \equiv 1 \pmod 4$. We shall consider hyperelliptic and non-hyperelliptic surfaces in different cases.

(i) Hyperelliptic case

In this section our aim is to study the automorphism groups of hyperelliptic M-surfaces with the M-property. As we shall show, we need to find possible extensions of $\theta_1 \colon \Lambda_1 \to C_{g+1}$ from NEC groups, which contain Λ_1, to finite groups, which contain C_{g+1}.

It follows from [3] that the group Λ_1, which has signature

$$(0; +; [(g+1)^{(2)}]; \{(\ \ )\}),$$

is always contained as a normal subgroup of index four in an NEC group Δ_1 with signature

$$(0;+;[\ ];\{(2^{(3)},g+1)\}).$$

We can extend $\theta_1\colon\Lambda_1\to C_{g+1}$ to an epimorphism $\mu_1\colon\Delta_1\to C_2\times D_{g+1}$ as follows.

Δ_1 and $C_2\times D_{g+1}$ have presentations

$$\Delta_1:\ \langle c_0,c_1,c_2,c_3\mid c_0^2=c_1^2=c_2^2=c_3^2=(c_0c_1)^2=(c_1c_2)^2=(c_2c_3)^2=(c_3c_0)^{g+1}=1\rangle$$

$$C_2\times D_{g+1}:\ \langle x,y,z\mid x^2=y^2=z^{g+1}=(xy)^2=xzxz^{-1}=(yz)^2=1\rangle.$$

Let us define $\mu_1\colon\Delta_1\to C_2\times D_{g+1}$ as follows:

$$\mu_1:\begin{cases} c_0 & \mapsto y\\ c_1 & \mapsto x\\ c_2 & \mapsto 1\\ c_3 & \mapsto yz.\end{cases}$$

Now let us calculate the signature of $Ker\mu_1$ using the Hoare's theorem.

$$\begin{aligned} C_2\times D_{g+1} \;=\; & \{1\}\cup\{x\}\cup\{y\}\cup\{z\}\cup\{xy\}\cup\{xz\}\cup\{yz\}\cup\{xyz\}\\ &\cup\{z^2\}\cup\{xz^2\}\cup\{yz^2\}\cup\{xyz^2\}\cup\{z^3\}\cup\{xz^3\}\cup\{yz^3\}\\ &\cup\{xyz^3\}\cup\cdots\cup\{z^{g-1}\}\cup\{xz^{g-1}\}\cup\{yz^{g-1}\}\\ &\cup\{xyz^{g-1}\}\cup\{z^g\}\cup\{xz^g\}\cup\{yz^g\}\cup\{xyz^g\}\end{aligned}$$

We have $4g+4$ cosets and let us label them as follows: $\{1\}=1$, $\{x\}=2$, $\{y\}=3$, $\{z\}=4$, $\{xy\}=5$, $\{xz\}=6$, $\{yz\}=7$, $\{xyz\}=8$, $\{z^2\}=9$, $\{xz^2\}=10$, $\{yz^2\}=11$, $\{xyz^2\}=12$, $\{z^3\}=13$, $\{xz^3\}=14$, $\{yz^3\}=15$, $\{xyz^3\}=16$, $\ldots$, $\{z^{g-1}\}=4g-3$, $\{xz^{g-1}\}=4g-2$, $\{yz^{g-1}\}=4g-1$, $\{xyz^{g-1}\}=4g$, $\{z^g\}=4g+1$, $\{xz^g\}=4g+2$, $\{yz^g\}=4g+3$, $\{xyz^g\}=4g+4$.

The action of the generators of Δ_1 on the cosets is given below.

$$\mu_1:\begin{cases} c_0 & \mapsto y\mapsto (1,3)(2,5)(4,4g+3)(6,4g+4)(7,4g+1)(8,4g+2)\\ & \qquad (9,4g-1)(10,4g)(11,4g-3)(12,4g-2)\cdots\\ c_1 & \mapsto x\mapsto (1,2)(3,5)(4,6)(7,8)(9,10)\cdots(4g+3,4g+4)\\ c_2 & \mapsto 1\mapsto (1)(2)(3)\cdots(4g+3)(4g+4)\\ c_3 & \mapsto yz\mapsto (1,7)(2,8)(3,4)(5,6)(9,4g+3)(10,4g+4)(11,4g+1)\cdots\end{cases}$$

The reflection c_2 fixes all the cosets while the others fix no cosets. So $Ker\mu_1$ will have $4g+4$ reflection generators. As usual, we call them $c_{21}, c_{22}, c_{23}, \ldots, c_{2,4g+4}$. The orbits of the dihedral group $\langle c_0, c_1\rangle \simeq D_2$ are $\{1,5\}$, $\{2,3\}$, $\{4,4g+4\}$, $\ldots$, $\{4g+3,6\}$. Since these orbits do not contain cosets fixed by c_0 and c_1 they induce no links. They induce only proper periods one. The orbits of the dihedral group $\langle c_1, c_2\rangle \simeq D_2$ are $\{1,2\}$, $\{3,5\}$, $\{4,6\}$, $\{7,8\}$, $\ldots$, $\{4g+1,4g+2\}$, $\{4g+3,4g+4\}$. In this case, each orbit contains two cosets fixed by c_2. Then $\{1,2\}$ induces the link $c_{21} \sim c_{22}$, $\{3,5\}$ induces the link $c_{23} \sim c_{25}$, ... and so on. Therefore, we get the following links: $c_{21} \sim c_{22}$, $c_{23} \sim c_{25}$, $c_{24} \sim c_{26}$, $c_{27} \sim c_{28}$, ... The orbits of the dihedral group $\langle c_2, c_3\rangle \simeq D_2$ are $\{1,7\}$, $\{2,8\}$, $\{3,4\}$, $\{5,6\}$, $\{9,4g+3\}$, $\{10,4g+4\}$, $\{11,4g+1\}$ .... Similarly, from these orbits we get the following links: $c_{21} \sim c_{27}$, $c_{22} \sim c_{28}$, $c_{23} \sim c_{24}$, $c_{25} \sim c_{26}$, $c_{29} \sim c_{2,4g+3}$, $c_{2,10} \sim c_{2,4g+4}$, $c_{2,11} \sim c_{2,4g+1}$ .... The orbits of the dihedral group $\langle c_3, c_0\rangle \simeq D_{g+1}$ are $\{1,4g+1,4g-3,4g-7,4g-11,\ldots\}$, and $\{2,4g+2,4g-2,4g-6,4g-10,\ldots\}$. Since these orbits do not contain cosets fixed by c_0 and c_3 they induce no links. They induce only proper periods one.

If we combine all these links we get the following chains: $c_{21} \sim c_{22} \sim c_{28} \sim c_{27} \sim c_{21}$, $c_{23} \sim c_{25} \sim c_{26} \sim c_{24} \sim c_{23}$, $c_{29} \sim c_{2,10} \sim c_{2,4g+4} \sim c_{2,4g+3} \sim c_{29}$, ... In total we get $g+1$ chains. Therefore, there are $g+1$ empty period cycles in the signature of $Ker\mu_1$. By the partition

$$A = \{1,4,5,8,9,12,\ldots,4g+1,4g+4\}$$

and

$$B = \{2,3,6,7,10,11,\ldots,4g-1,4g+2,4g+3\}$$

we see that $\mathcal{U}/Ker\mu_1$ is orientable. Note that

$$A = \{n \,|n \in \mathbf{N},\ 1 \le n \le g+1,\ n \equiv 0(\mathrm{mod}\,4) \text{ or } n \equiv 1(\mathrm{mod}\,4)\}$$

and

$$B = \{n \,|n \in \mathbf{N},\ 1 \le n \le g+1,\ n \equiv 2(\mathrm{mod}\,4) \text{ or } n \equiv 3(\mathrm{mod}\,4)\},$$

where $\mathbf{N}$ is the set of natural numbers.

By the Riemann-Hurwitz formula we find that the genus is 0 and finally the signature of $Ker\mu_1$ is

$$(0;+;[\,];\{(\quad)^{(g+1)}\}).$$

Note that the restriction of μ_1 to Λ_1 is θ_1. This is because there is a unique epimorphism from Λ_1 to C_{g+1} whose kernel has signature $(0;+;[\,];\{(\ \ )^{(g+1)}\})$, that is, if there is another such epimorphism then they differ by an automorphism of C_{g+1}. So $\mathcal{U}/Ker\mu_1$ is a Klein surface of genus 0 with $g+1$ boundary components and its automorphism group is isomorphic to $C_2 \times D_{g+1}$. Its complex double X is a Riemann surface of genus g with the M-property and $C_2 \times C_2 \times D_{g+1} \subset AutX$, where $AutX$ denotes the full automorphism group of X including the anticonformal ones. We can easily see this by defining an epimorphism $\theta^*: \Delta_1 \rightarrow C_2 \times C_2 \times D_{g+1}$ by means of $\mu_1: \Delta_1 \rightarrow C_2 \times D_{g+1}$, where $Ker\theta^*$ is a Fuchsian group with signature $(g;-)$ and $\mathcal{U}/Ker\theta^*$ is conformally equivalent to X.

$C_2 \times C_2 \times D_{g+1}$ has a presentation

$$\langle k, x, y, z \mid k^2 = x^2 = y^2 = z^{g+1} = (kx)^2 = (ky)^2 = kzkz^{-1} = (xy)^2 = \\ = xzxz^{-1} = (yz)^2 = 1\rangle$$

Let us define $\theta^*: \Delta_1 \rightarrow C_2 \times C_2 \times D_{g+1}$ as follows:

$$\theta^* : \begin{cases} c_0 & \mapsto ky \\ c_1 & \mapsto kx \\ c_2 & \mapsto k \\ c_3 & \mapsto kyz. \end{cases}$$

Then, we see that θ^* is an extension of an epimorphism $\theta: \Delta_1^+ \rightarrow C_2 \times D_{g+1}$, where Δ_1^+ is the canonical Fuchsian group of Δ_1, so it has signature $[2^{(3)}, g+1]$.

Δ_1^+ and $C_2 \times D_{g+1}$ have the following presentations:

$$\Delta_1^+ : \ \langle u_1, u_2, u_3, u_4 \mid u_1^2 = u_2^2 = u_3^2 = u_4^{g+1} = u_1u_2u_3u_4 = 1\rangle,$$

$$C_2 \times D_{g+1} : \ \langle x, y, z \mid x^2 = y^2 = z^{g+1} = (xy)^2 = xzxz^{-1} = (yz)^2 = 1\rangle.$$

Note that if we take $u_1 = c_0c_1$, $u_2 = c_1c_2$, $u_3 = c_2c_3$, $u_4 = c_3c_0$, then we see that each u_i $(i = 1, 2, 3, 4)$ satisfies the conditions in the presentation of Δ_1^+, where c_is $(i = 0, 1, 2, 3)$ are the generators of Δ_1. Now we can define $\theta: \Delta_1^+ \rightarrow C_2 \times D_{g+1}$ by means of $\theta^*: \Delta_1 \rightarrow C_2 \times C_2 \times D_{g+1}$ as follows:

$$\theta : \begin{cases} u_1 & \mapsto yx \\ u_2 & \mapsto x \\ u_3 & \mapsto yz \\ u_4 & \mapsto z^{-1}. \end{cases}$$

Since θ preserves the orders of the generators of Δ_1^+ we can easily see that $Ker\theta$ is a Fuchsian group with signature $(g;-)$. Also we can see that θ is the restriction of θ^* to Δ_1^+ and so θ and θ^* have the same kernel.

We can show that $(\theta^*)^{-1}(\langle k\rangle)$ has signature

$$(0;+;[\ ];\{(\quad)^{(g+1)}\})$$

and hence k is an M-symmetry.

As we mentioned earlier, all M-surfaces arising in this way are hyperelliptic and now we will show this. We know that $\theta\colon \Delta_1^+ \to C_2 \times D_{g+1}$ is a surface kernel epimorphism and the generator of C_2, x, is a central element in the conformal automorphism group of the surface $X = \mathcal{U}/Ker\theta$. By using the Hoare's theorem we can show that $\theta^{-1}(\langle x\rangle)$ is a Fuchsian group with signature $[2^{(2g+2)}]$. This means x is the hyperelliptic involution and hence the corresponding Riemann surface $X = \mathcal{U}/Ker\theta$ is hyperelliptic.

Since we shall show that the epimorphisms θ_2 and θ_3 yield non-hyperelliptic surfaces than we have:

Theorem 3.3. *Let $X = \mathcal{U}/K$ be a hyperelliptic M-surface of genus $g > 1$ with the M-property. Then K is always contained as a normal subgroup of index $8g+8$ in an NEC group Δ, where Δ has signature $(0,+,[-],\{(2^{(3)},g+1)\})$ and Δ/K is isomorphic to $C_2\times C_2\times D_{g+1}$ and contained in $Aut^{\pm}(X)$.*

■

Geometrically, we can construct a hyperelliptic M-surface with the M-property as follows. Choose a right rectangular geodesic $(2g+2)$-gon in the hyperbolic plane and label its sides by the integers from 1 to $2g+2$ following the positive orientation. Assume that the even sides and the odd sides have all the same length. Take a second copy of the $(2g+2)$-gon and identify either the even or the odd sides of the first polygon with the corresponding ones of the second. Then we obtain a sphere with $g+1$ holes, which is a Klein surface of genus 0 with $g+1$ boundary components. Take the complex double of this Klein surface. Then we get a Riemann surface of genus g with the M-property. See Figure 1.

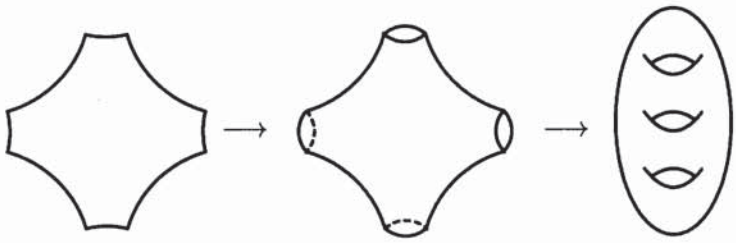

Figure 1

Remark 3.4. In the geometric construction above, if we begin with a regular $(2g+2)$-gon then we get a Riemann surface Y of genus g admitting a conformal automorphism of order $2g+2$ which fixes the centres of four $(2g+2)$-gons. It follows from Accola [1] and Maclachlan [8] that Y is the Accola-Maclachlan surface of genus g. It is a *Platonic* surface and is uniformised by a normal subgroup of a Fuchsian group of signature $[2, 4, 2g+2]$. By a Platonic surface we mean a surface that is uniformised by a normal subgroup of a Fuchsian group of signature $[2, m, n]$, where $1/m + 1/n < 1/2$. Thus, Y is a Platonic M-surface of genus g with the M-property. In [9] we showed that for every $g > 1$ the Accola-Maclachlan surface is the only Platonic M-surface of genus g.

In Theorem 3.3, $C_2 \times D_{g+1}$ is the full conformal automorphism group of a hyperelliptic M-surface of genus g with the M-property except in the case where the surface is Platonic. Remark that NEC groups with signature $(0, +, [-], \{(2^{(3)}, g+1)\})$ are only properly contained in triangular NEC groups. Thus the exception is for Y, the Accola-Maclachlan surface of genus g. Y admits $8g+8$ conformal automorphisms and Aut^+Y has a presentation

$$\langle A, B \mid A^4 = B^{(2g+2)} = (AB)^2 = (A^{-1}B)^2 = 1 \rangle.$$

(For details about these surfaces see Accola [1] and Maclachlan [8]).

(ii) Non-hyperelliptic case

As in the hyperelliptic case our aim in this section is to study the automorphism groups of non-hyperelliptic M-surfaces with the M-property. As we shall show, we need to find possible extensions of $\theta_2 \colon \Lambda_2 \to C_{g+1}$ and $\theta_3 \colon \Lambda_3 \to C_{g+1}$ from NEC groups, which contain Λ_2 and Λ_3, to finite groups, which contain C_{g+1}.

Using the list in [3], we see that Λ_2, which has signature

$$(0; +; [\frac{g+1}{2}]; \{(\ )^{(2)}\}),$$

is always contained as a normal subgroup of index two in an NEC group Δ_2 with signature

$$(0; +; [\ ]; \{(2^{(4)}, \frac{g+1}{2})\}).$$

Similarly, by the list of [3], Λ_3, which has signature

$$(1; -; [\frac{g+1}{2}]; \{(\ )\}),$$

is always contained as a normal subgroup of index two in an NEC group Δ_3 with signature

$$(0; +; [2]; \{(2^{(2)}, \frac{g+1}{2})\}).$$

In general, Δ_2 and Δ_3 are not contained in any other NEC groups. However, there are special cases where the NEC groups Δ_2 and Δ_3 are contained in some other NEC groups.

Now, we are looking for finite groups containing C_{g+1} with index two such that we can extend $\theta_2: \Lambda_2 \to C_{g+1}$ and $\theta_3: \Lambda_3 \to C_{g+1}$ to epimorphisms from Δ_2 and Δ_3 to these finite groups. Let G be such a group. Suppose that $\mu_i: \Delta_i \to G$ $(i = 2, 3)$ is an epimorphism which extends $\theta_i: \Lambda_i \to C_{g+1}$ $(i = 2, 3)$. Then G will be the automorphism group of a Klein surface, say S, of genus 0 with $g+1$ boundary components. As we shall see later, the complex double of S is a non-hyperelliptic M-surface, say X, with the M-property. By Theorem 2.1, $AutX \simeq C_2 \times Aut^+X$, where C_2 is generated by the M-symmetry and Aut^+X is isomorphic to a subgroup of the rotation group of the sphere. On the other hand, we know that the automorphism group of S consists of conformal automorphisms of X commuting with the M-symmetry, see [2, Theorem 1.11.1]. Therefore, $Aut^+X \simeq G$ and G must be a subgroup of the rotation group of the sphere. It is well-known that any finite rotation group of the sphere is cyclic, dihedral, or isomorphic to A_4, S_4 or A_5. Therefore, G can only be isomorphic to C_{2g+2} or D_{g+1}. Since the kernel of the epimorphism that we are looking for is a surface group, then we cannot define an epimorphism from Δ_2 (or Δ_3) to C_{2g+2} as required. So the only possibility is that $G \simeq D_{g+1}$.

Having found the groups containing Λ_i $(i = 2, 3)$ and C_{g+1}, we can now define the epimorphisms. Let us begin with Δ_2.

Δ_2, which has signature $(0; +; [\ ]; \{(2^{(4)}, \frac{g+1}{2})\})$, and D_{g+1} have the following presentations:

$$\Delta_2 : \langle c_0, c_1, c_2, c_3, c_4 \mid c_0^2 = c_1^2 =$$

$$c_2^2 = c_3^2 = c_4^2 = (c_0c_1)^2 = (c_1c_2)^2 = (c_2c_3)^2 = (c_3c_4)^2 = (c_4c_0)^{\frac{g+1}{2}} = 1\rangle$$

$$D_{g+1} : \langle x, y \mid x^2 = y^{g+1} = (xy)^2 = 1\rangle.$$

Let us define $\mu_2 \colon \Delta_2 \to D_{g+1}$ as follows

$$\mu_2 : \begin{cases} c_0 & \mapsto x \\ c_1 & \mapsto 1 \\ c_2 & \mapsto x \\ c_3 & \mapsto y^{\frac{g+1}{2}} \\ c_4 & \mapsto xy^{-2}. \end{cases}$$

Note that unless $\frac{g+1}{2}$ is odd, 2 and $\frac{g+1}{2}$ are not coprime and hence μ_2 is not an epimorphism and that μ_2 is an extension of θ_2. Similarly, using the Hoare's theorem we can find that $Ker\mu_2$ has signature

$$(0; +; [\ ]; \{(\ \ )^{(g+1)}\}).$$

Our aim now is to extend $\theta_3 \colon \Lambda_3 \to C_{g+1}$ to an epimorphism $\mu_3 \colon \Delta_3 \to D_{g+1}$ such that $Ker\mu_3$ has signature

$$(0; +; [\ ]; \{(\ \ )^{(g+1)}\}).$$

Δ_3, which has signature $(0; +; [2]; \{(2^{(2)}, \frac{g+1}{2})\})$, and D_{g+1} have the following presentations:

$$\Delta_3 : \langle u, c_0, c_1, c_2, c_3 \mid u^2 = c_0^2 = c_1^2 = c_2^2 = c_3^2 = (c_0c_1)^2 = (c_1c_2)^2 =$$

$$(c_2c_3)^{\frac{g+1}{2}} = uc_0uc_3 = 1\rangle$$

$$D_{g+1} : \langle x, y \mid x^2 = y^{g+1} = (xy)^2 = 1\rangle.$$

Let us define $\mu_3 \colon \Delta_3 \to D_{g+1}$ as follows

$$\mu_3 : \begin{cases} u & \mapsto x \\ c_0 & \mapsto xy \\ c_1 & \mapsto 1 \\ c_2 & \mapsto xy \\ c_3 & \mapsto yx. \end{cases}$$

As before, using the Hoare's theorem we can find that $Ker\mu_3$ has signature

$$(0;+;[\ ];\{(\quad)^{(g+1)}\}).$$

Note that θ_i $(i = 2,3)$ is the restriction of μ_i by construction.

We now show that those M-surfaces corresponding to μ_2 and μ_3 are non-hyperelliptic. As we know, the hyperelliptic involution is central in the automorphism group. Since g is odd, $g+1$ is even and there is only one central element in D_{g+1}, which is $y^{\frac{g+1}{2}}$ and has order 2. However, using the Hoare's theorem we can find that $\mu_i^{-1}(< y^{(g+1)/2} >)$ has signature different from $(0,+,[-],\{(2^{(2g+2)})\})(i = 2,3)$. Therefore, these surfaces are non-hyperelliptic.

For non-hyperelliptic M-surfaces we can summarise our results in the following theorem.

Theorem 3.5. *Let X be a non-hyperelliptic M-surface of genus $g > 1$ (g odd) with the M-property and $T : X \to X$ be the M-symmetry. If $X/ < T >= U/\Gamma$ then Γ is always contained as a normal subgroup of index $2g+2$ in an NEC group Δ, where Δ has signature $(0,+,[-],\{(2^{(4)},(g+1)/2)\})$ and Δ/Γ is isomorphic to D_{g+1} and contained in $Aut(X/ < T >)$.*

■

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