Some remarks on the Backus problem in Geodesy

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Abstract

This paper is devoted to the Backus problem, a boundary value problem for the Laplace equation on the exterior of a bounded open set with a fully nonlinear boundary condition. We get a new uniqueness theorem for this problem and propose a way of proving existence of solutions.

1 Introduction

In this paper we deal with the Backus problem:

$$\begin{cases} \Delta u = 0 & \textit{outside } S, \\ |\nabla u| = g & \text{on } S, \\ u(x) \to 0 & \text{as } x \to \infty, \end{cases} \tag{1}$$

where S is a closed surface in $\mathbb{R}^3$, Ω denotes its exterior domain and g is a given positive continuous function on S.

The physical motivation for this problem comes from Geodesy (see, for example, [2],[6],[11],[12],[17], [30]) and Geomagnetism (see, [2],[3],[4] and [19]). Assuming to be known the surface of the Earth S, in Geodesy it is posed the problem of whether the *external* gravitational field of the Earth can be (or not) determined merely from measurements of its intensity on the Earth surface. If by u we denote the gravitational or newtonian potential of the Earth, and g denotes the modulus of the force of gravity on S (in Geodesy g is simply called *gravity*), then by well known properties of u (see, for example, [13, Chapter 1]) the problem posed above leads to a boundary problem like (1). (We observe that this is only true if we do not take into consideration the Earth rotation as we shall assume here; for a more complete model see, for example, [6]). This geodetic problem is quite realistic since the gravity can be easily measured both in land and sea, and by spatial positioning techniques the hypothesis concerning the knowledge of S is not far from be realistic too nowdays. In Geomagnetism we may formulate a completely analogous problem for the external magnetic field of the Earth.

In $\mathbb{R}^2$, a problem like (1), where now S is a simple closed curve, can be essentially reduced to solve a Dirichlet problem (see [9] for a physical motivation in dimension two where however u is harmonic in the interior of S). In fact, if without loss of generality by Riemann's mapping theorem, S is the unit circle and we replace the condition at ∞ in (1) by *u bounded in Ω*, then by the inversion defined by $z = x+iy \mapsto \bar{z}^{-1}$ where $\bar{z} = x - iy$ and $(x,y) \in \mathbb{R}^2$, the function $\tilde{u}(z) := u(1/\bar{z})$ is harmonic in the interior D of the unit disk and, in addition, $|\nabla\tilde{u}| = |\nabla u|$ on $|z| = 1$. Now, the function $f(z) = \partial\tilde{u}/\partial x - i\partial\tilde{u}/\partial y$ is analytic in D and, if $f(z) \neq 0$, then $\log f(z)$ is harmonic as well. The real part of $\log f(z)$ is precisely $\log|\nabla\tilde{u}|$ and this proves that $\log|\nabla\tilde{u}|$ is harmonic in D. In this way, if g is strictly positive, all we have to do is to solve the Dirichlet problem

$$\Delta v = 0 \quad \text{in } D, \quad v = \log g \quad \text{on } |z| = 1.$$

where $v = \log|\nabla\tilde{u}|$. By means of the Cauchy-Riemann equations we then may determine $\log f(z)$ and hence $f(z)$. This is the strategy to solve in $\mathbb{R}^2$ the problem we are concerned with. Nevertheless the solution is not unique, since there are infinitely many harmonic gradient vector fields $(\partial\tilde{u}/\partial x, \partial\tilde{u}/\partial y)$ whose modulus take the same value on $|z| = 1$ (for more details, see [2, Theorem 1] and [22]; in [22] the case where g may vanish is also considered).

In $\mathbb{R}^3$ this approach is not feasible and neither the inversion map preserves harmonicity nor $\log|\nabla u|$ is harmonic if u is harmonic. The first drawback could be overcome by considering the Kelvin transformation (see [1, Chapter 4]); but if we do so the boundary condition is not preserved and it changes slightly. For example, if now v denotes the Kelvin transform of u and

$$S = \{\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}, \tag{2}$$

then it can be proved (see [24]) that (1) is equivalent to

$$\begin{cases} \Delta v = 0 & \text{inside } S, \\ \left(v + \dfrac{\partial v}{\partial \mathbf{n}}\right)^2 + |\nabla_s v|^2 = g^2 & \text{on } S, \end{cases} \tag{3}$$

where $\partial v/\partial \mathbf{n}$ is the outer normal derivative of v and $\nabla_s v$ denotes the tangential or surface gradient of v.

At this point it should be noted that the problem

$$\Delta u = 0 \quad \text{in } D, \quad |\nabla u| = g \text{ on } S, \tag{4}$$

where D is the *interior* of a closed surface S has no special relevance in Geodesy at best of our knowledge, and it is completely different to (1) (see Remark 2.3 in Section 2). This interior problem has been studied by some authors (see, for example, [20] and [21]).

As far as the authors know there is not yet a *global* existence theorem for (1). Some local existence theorems are known (see [6], [15] and [29]): roughly speaking, *if g is close enough* (in a convenient Hölder space of functions) *to some g_0 such that $g_0 = |\nabla u_0|$ on S, where u_0 is a given, regular at infinity, harmonic function in Ω, then there is a function u close to u_0 solution of* (1). (Hereafter, by a solution of (1) we mean a function $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$, vanishing at infinity, and satisfying pointwise both the Laplace equation and the boundary condition).

The following uniqueness result for problem (1) is well known (see [2], [17]): *there is at most one solution of* (1) *whose normal derivative is strictly negative* (*or strictly positive*) *at each point of S*. In Section 2 of this paper we generalize this result to functions with *nonpositive* (or *nonnegative*) normal derivative. Although our approach is the same as that followed by Backus (via the maximum principle), our generalization comes from a slightly more careful examination of the boundary condition.

In Section 3 we present a possible way of proving existence of solutions of Problem (3). The key idea of the approach we propose is based in the following simple remark: if (3) has a solution which satisfies $v + \partial v/\partial \mathbf{n} \geq 0$ then necessarily

$$v + \frac{\partial v}{\partial \mathbf{n}} = \sqrt{g^2 - |\nabla_s v|^2}\,.$$

This leads to the consideration of the associated *oblique* nonlinear boundary problem (see (7)). In Section 3 we state a uniqueness theorem for this problem (7) and study the relationship between this problem and the problem (3).

Throughout this paper we shall denote by $\mathcal{H}(\Omega)$ the real space of harmonic functions in an open subset Ω of $\mathbb{R}^N$. For unbounded Ω, $\mathcal{H}_\infty(\Omega)$ will denote the subset of $\mathcal{H}(\Omega)$ consisting of functions vanishing at infinity. If S is a closed surface in $\mathbb{R}^3$, we shall use the notation $C_+(S) = \{g \in C(S) : g(x) \geq 0 \, \forall x \in S\}$.

2 Some results about the uniqueness of solutions

In this Section we shall often use the following

Lemma 2.1 *Let S be a closed surface in $\mathbb{R}^3$ and let Ω be the unbounded connected component of $\mathbb{R}^3 \setminus S$. Let $u \in \mathcal{H}_\infty(\Omega) \cap C^1(\bar{\Omega})$, $u \not\equiv 0$, be such that $\lim_{x\to\infty} u(x) = 0$. Then*

$$\min(m, 0) < u(x) < \max(M, 0), \qquad \forall x \in \Omega$$

where $m = \min_S u$ and $M = \max_S u$.

Proof. Without loss of generality we can assume that $0 \notin \bar{\Omega}$. Let $B(0, R)$ denote an open ball centered at the origin of radius R and containing S. Since $\Omega_R = \Omega \cap B(0, R)$ is connected and u is not constant in Ω_R by real analyticity of harmonic functions (see, for example, [1, Theorem 1.24], then by the maximum principle for harmonic functions (see [7, §2.2, Corollaire 3]) we have

$$\min_{\partial\Omega_R} u < u(x) < \max_{\partial\Omega_R} u \qquad \forall x \in \Omega_R\,,$$

where $\partial\Omega_R = S \cup \partial B(0,R)$. In addition,

$$\min_{\partial\Omega_R} u = \min(m, m(R)) \quad \text{and} \quad \max_{\partial\Omega_R} u = \max(M, M(R))\,,$$

where

$$m(R) = \min_{\partial B(0,R)} u \quad \text{and} \quad M(R) = \max_{\partial B(0,R)} u\,.$$

Letting $R \to \infty$ and observing that both $m(R)$ and $M(R)$ converge to zero, we obtained the desired result. □

We also recall the *Hopf boundary point lemma* (see [10, Lemma 3.4])

Lemma 2.2 *Let Ω be a domain in $\mathbb{R}^N$ and $u \in \mathcal{H}(\Omega)$. Let $x_0 \in \partial\Omega$ be such that*

(a) *u is continuous at x_0 ;*

(b) *$u(x_0) > u(x)$ for all $x \in \Omega$;*

(c) *$\partial\Omega$ satisfies an interior sphere condition at x_0 .*

Then, the outer normal derivative of u at x_0, if it exists, satisfies the strict inequality

$$\frac{\partial u}{\partial \mathbf{n}}(x_0) > 0\,.$$

We are now in position of proving our first theorem. We shall assume that S is regular enough as to apply Lemma 2.2, and by $\partial/\partial\mathbf{n}$ we shall mean the derivative along the normal of S pointing to the *exterior* of S.

Theorem 2.3 *Let $u, v \in \mathcal{H}_\infty(\Omega) \cap C^1(\bar{\Omega})$ be such that*

(a) *$\partial u/\partial\mathbf{n}, \partial v/\partial\mathbf{n} \leq 0$ on S;*

(b) *$|\nabla u| = |\nabla v|$ on S.*

Then $u \equiv v$.

Proof. Since $|\nabla u| = |\nabla v|$ on S, then on S we have

$$\left(\frac{\partial u}{\partial \mathbf{n}}\right)^2 + |\nabla_s u|^2 = \left(\frac{\partial v}{\partial \mathbf{n}}\right)^2 + |\nabla_s v|^2 \,,$$

where $\nabla_s u$ and $\nabla_s v$ are the surface gradients of u and v respectively. Let $w = u - v$, and define $m = \min_S w$ and $M = \max_S w$. Let x_0, $\tilde{x}_0 \in S$ be such that $M = w(x_0)$ and $m = w(\tilde{x}_0)$. Since $w \in C^1(\bar{\Omega})$ then $\nabla_s w$ must vanish at x_0 and $\tilde{x}_0$. Then at these points we have

$$\left(\frac{\partial u}{\partial \mathbf{n}}\right)^2 = \left(\frac{\partial v}{\partial \mathbf{n}}\right)^2 \,,$$

and since $\partial u/\partial \mathbf{n}, \partial v/\partial \mathbf{n} \leq 0$, then at both x_0 and $\tilde{x}_0$ we have

$$\frac{\partial w}{\partial \mathbf{n}} = 0 \,. \tag{5}$$

Now we can prove that $w \equiv 0$. If w does not vanish identically, then by Lemma 2.1 we have

$$\min(m, 0) < w(x) < \max(M, 0) \qquad \forall x \in \Omega \,.$$

If $M \geq 0$ then $w(x) < w(x_0)$ for all $x \in \Omega$, and by Hopf's Lemma we have $\partial w/\partial \mathbf{n}(x_0) < 0$, contradicting the hypothesis (5). If on the contrary $M < 0$ then $m < 0$ and $w(\tilde{x}_0) < w(x)$, so by Hopf's Lemma we infer that $\partial w/\partial \mathbf{n}(\tilde{x}_0) > 0$, contradicting again (5). □

Remark 2.1 Alternatively, we can prove that there is at most one solution of (1) whose normal derivative is nonnegative. □

Remark 2.2 Compare this Theorem 2.3 with [25, Theorem 1(α)] and [26, Corollary 1], where uniqueness results have been obtained for the problems

$$\Delta u = -2 \quad \text{in } D \subset \mathbb{R}^N, \quad |\nabla u| = g \geq 0 \text{ on } \partial D,$$

and

$$\Delta u = f(u) \quad \text{in } D \subset \mathbb{R}^N, \quad |\nabla u| = g \geq 0 \text{ on } \partial D,$$

where D is a bounded domain in $\mathbb{R}^N$ and f satisfies

$$f'(s) \geq 0 \quad (f'(s) \not\equiv 0), \quad f(0) = 0. \quad \square$$

Remark 2.3 Observe that for the interior problem (4) the assumption $\partial u/\partial n \le 0$ on S for the solution has no sense, since if $u \in \mathcal{H}(\Omega) \cap C^1(\bar{\Omega})$ then

$$\int_S \frac{\partial u}{\partial \mathbf{n}}\, ds = 0\,,$$

and so $\partial u/\partial \mathbf{n}$ necessarily changes sign on S unless u is constant. □

Example 2.1 Let S be the unit sphere in $\mathbb{R}^3$. Let c be an arbitrary positive constant. In this case, the functions $\pm c/r$, where $r = |x|$, are the radial solutions of (1). Let $u = c/r$. Since $\partial u/\partial r = -c < 0$ on S, then by Theorem 2.3 this u is the unique solution of (1) with $g = c$ which satisfies $\partial u/\partial \mathbf{n} \le 0$. □

Without any restriction on the sign of the normal derivative of the solution, it is clear that if u is a solution of (1) then $-u$ is a solution as well. We then could wonder if these functions u and $-u$ are the only solutions of the problem. In general the answer is negative as it was proved by Backus (see [3]). In fact, let $\widetilde{\mathcal{H}}_\infty(\Omega)$ be the subset of $\mathcal{H}_\infty(\Omega) \cap C^1(\bar{\Omega})$ consisting of functions z not vanishing identically and such that the oblique boundary value problem

$$\begin{cases} \Delta w = 0 & \text{outside } S, \\ \langle \nabla w, \nabla z \rangle = 0 & \text{on } S, \\ w(x) \to 0 & \text{as } x \to \infty, \end{cases} \tag{6}$$

has a nontrivial $C^2(\Omega) \cap C^1(\bar{\Omega})$ solution. Since $|\nabla u| = |\nabla v|$ if and only if

$$\langle \nabla(u - v), \nabla(u + v) \rangle = 0\,,$$

we then have the following

Proposition 2.4 *$\widetilde{\mathcal{H}}_\infty(\Omega) \ne \emptyset$ if and only if there exist two functions $u, v \in \mathcal{H}_\infty(\Omega) \cap C^1(\bar{\Omega})$ $(u \not\equiv \pm v)$ such that $|\nabla u| = |\nabla v|$ on S.*

Proof. Let $z \in \widetilde{\mathcal{H}}_\infty(\Omega)$ and let w be a nontrivial solution of (6). Define $u = (w + z)/2$ and $v = (w - z)/2$. Then $u, v \in \mathcal{H}_\infty(\Omega) \cap C^1(\bar{\Omega})$ and

$$|\nabla u|^2 = \frac{1}{2}\left(|\nabla w|^2 + |\nabla z|^2\right) = |\nabla v|^2$$

on S.

On the other hand, if u and v ($u \not\equiv \pm v$) are such that $|\nabla u| = |\nabla v|$ on S, then $u + v \in \widetilde{\mathcal{H}}_\infty(\Omega)$. This completes the proof. $\square$

In the case of a sphere, Backus proved (see [3]) that $\widetilde{\mathcal{H}}_\infty(\mathbb{R}^3 \setminus \bar{B}(0, R)) \neq \emptyset$. In fact he found non trivial solutions of (6) by choosing $z = x_3/r^3 \in \mathcal{H}_\infty(\mathbb{R}^3 \setminus \{0\})$. See [16] for a related topic.

Remark 2.4 If S is smooth enough, it should be observed that if $z \in \widetilde{\mathcal{H}}_\infty(\Omega)$ then ∇z is tangential to S in some set $T \subset S$. In fact, if $T = \emptyset$ then it follows that the only solution of (6) is $w = 0$ (see, for example, [23]). In the above example of Backus the tangential set T is the equator of the sphere. $\square$

Remark 2.5 The following question, posed by Backus ([3]), seems to remain open up to date: let $u \in \mathcal{H}_\infty(\Omega) \cap C^1(\bar{\Omega})$; how many functions v are there in $\mathcal{H}_\infty(\Omega) \cap C^1(\bar{\Omega})$ that have $|\nabla v| = |\nabla u|$ on S? $\square$

3 A possible way of proving existence

As we said in the Introduction, in this Section we propose an approach to prove an existence result for (1). We shall restrict ourselves to the simplest case of a sphere (2) and we consider the equivalent problem (3). The basic idea we propose is to consider the boundary value problem

$$\begin{cases} \Delta v = 0 & \text{in } \Omega = B(0,1), \\ v + \dfrac{\partial v}{\partial \mathbf{n}} = \sqrt{(g^2 - |\nabla_s v|^2)_+} & \text{on } \partial\Omega, \end{cases} \tag{7}$$

where

$$(g^2 - |\nabla_s v|^2)_+ = \max\{(g^2 - |\nabla_s v|^2), 0\}\,.$$

(Hereafter we shall exclude the case $g \equiv 0$, since if $g \equiv 0$, by Theorem 3.1, the only solution of Problem (7) is $v \equiv 0$). Our first result in this direction is the following uniqueness result for the problem (7):

Theorem 3.1 *The problem* (7) *has at most one solution* $v \in C^2(\Omega) \cap C^1(\bar{\Omega})$.

Proof. Let v and w be two solutions of (7) and let $z = v - w$. Since $z \in \mathcal{H}(\Omega) \cap C^1(\bar{\Omega})$, it takes its maximum value at some point $x_0 \in \partial\Omega$ and its minimum value at some point $\tilde{x}_0 \in \partial\Omega$. Moreover, $\nabla_s z = 0$ at x_0 and $\tilde{x}_0$, so it follows that at x_0 and $\tilde{x}_0$ we have

$$z + \partial z/\partial \mathbf{n} = 0\,.$$

Now we infer that $z(x_0) \leq 0$ and $z(\tilde{x}_0) \geq 0$, and this of course implies $z \equiv 0$. In fact, if $z(x_0) > 0$ then $\partial z/\partial \mathbf{n}(x_0) < 0$ but this is not possible at a maximum point; on the other hand, if $z(\tilde{x}_0) < 0$ then $\partial z/\partial \mathbf{n}(\tilde{x}_0) > 0$ which is not possible at a minimum point. This completes the proof of this Lemma. □

The relationship between the problems (7) and (3) is made clear in the following

Lemma 3.2 *Let v be the solution (assumed to exist) of* (7). *If*

$$|\nabla_s v| \leq g \qquad on\ \partial\Omega\,, \tag{8}$$

then v is the unique solution of (3) *such that $v + \partial v/\partial \mathbf{n} \geq 0$ on $\partial\Omega$.*

In addition, if v does not satisfy (8) *then the boundary value problem* (3) *has no solutions satisfying $v + \partial v/\partial \mathbf{n} \geq 0$ on $\partial\Omega$.* □

Remark 3.1 In the first part of this Lemma, the uniqueness of v is clear from Theorem 2.3. In fact, the function v in (3) is the Kelvin transform of u, that is to say

$$v(x) = \frac{1}{|x|} u\left(\frac{1}{|x|^2} x\right)\,;$$

so we have on $\partial\Omega$

$$v + \partial v/\partial \mathbf{n} = -\partial u/\partial \mathbf{n}\,,$$

and then $v + \partial v/\partial \mathbf{n} \geq 0$ if and only if $\partial u/\partial \mathbf{n} \leq 0$. □

Remark 3.2 For an arbitrary positive constant c, if $g(x) \equiv c$ then $v \equiv c$ is the unique solution of (7). Since $\nabla_s v \equiv 0$, by Lemma 3.2 we then conclude that $v \equiv c$ is the unique solution of (3) satisfying $v + \partial v/\partial \mathbf{n} \geq 0$ on $\partial\Omega$. Compare this result with Example 2.1. □

Therefore, what we want to prove is that indeed the problem (7) has a solution and we have (8) for the solution of (7). Then we could conclude an existence theorem for (3). We have still not proved these things but we state the following

Conjeture. *The problem* (7) *has a unique solution* $v \in C^2(\Omega) \cap C^1(\bar{\Omega})$. *In addition, v satisfies* (8).

This conjecture is based on the remainder results of this Section and on some additional work of the authors (see [8]).

With respect to the condition (8) we have the following result:

Proposition 3.3 *Let* $g \in C_+(\partial\Omega)$ *and let v be a classical solution (assumed to exist) of* (7). *Then*

$$\{x \in \partial\Omega : |\nabla_s v| < g\} \neq \emptyset\,.$$

Proof. Assumed on the contrary that $|\nabla_s v| \geq g$ on $\partial\Omega$. Then we have

$$v + \partial v / \partial \mathbf{n} = 0 \quad \text{on } \partial\Omega.$$

With the same argument used in the proof of Theorem 3.1 we now conclude that $v \equiv 0$ in $\bar{\Omega}$, and this would imply that $g \equiv 0$. The proof is complete. □

Remark 3.3 If $g > 0$ the conclusion of this proposition directly follows from the fact that if $v \in C^1(\bar{\Omega})$, as we are assuming, then the tangential gradient of v vanish at the points of the boundary where the harmonic function v reaches its maximum and minimum values. □

We now introduce the following sets

$$A_- = \{x \in \partial\Omega : |\nabla_s v| < g\}$$

and

$$A_+ = \{x \in \partial\Omega : |\nabla_s v| > g\}\,.$$

In order to obtain some more information about these sets, we shall use the following identity which can be infered from an integral identity due to F.Rellich ([28]); see also [27, (2.14)]:

Proposition 3.4 *Let* $v \in \mathcal{H}(\Omega) \cap C^1(\bar{\Omega})$, *where* $\Omega = B(0,1)$ *in* $\mathbb{R}^N$ $(N \geq 2)$. *Then,*

$$(N-2)\int_\Omega |\nabla v|^2\, dx = \int_{\partial\Omega} \left(|\nabla_s v|^2 - \left|\frac{\partial v}{\partial \mathbf{n}}\right|^2 \right) ds\,. \quad \square$$

Since by Green's first identity we have

$$\int_\Omega v\Delta v\,dx + \int_\Omega |\nabla v|^2\,dx = \int_{\partial\Omega} v\frac{\partial v}{\partial \mathbf{n}}\,ds\,,$$

if $v \in \mathcal{H}(\Omega)$, then

$$\int_{\partial\Omega} v\frac{\partial v}{\partial \mathbf{n}}\,ds = \int_\Omega |\nabla v|^2\,dx \geq 0\,.$$

By Proposition 3.4 we then have ($\mathrm{N} > 2$),

$$\int_{\partial\Omega} v\frac{\partial v}{\partial \mathbf{n}}\,ds = \frac{1}{\mathrm{N}-2}\int_{\partial\Omega}\left(|\nabla_s v|^2 - \left|\frac{\partial v}{\partial \mathbf{n}}\right|^2\right)\,ds\,. \tag{9}$$

Since on the other hand we can write

$$\begin{aligned}\int_{\partial\Omega}\left(v\frac{\partial v}{\partial \mathbf{n}} + \left|\frac{\partial v}{\partial \mathbf{n}}\right|^2\right)\,ds &= \int_{\partial\Omega}\left[\left(v + \frac{\partial v}{\partial \mathbf{n}}\right)^2 ds - v^2 - v\frac{\partial v}{\partial \mathbf{n}}\right]\,ds\\ &\leq \int_{\partial\Omega}\left(v + \frac{\partial v}{\partial \mathbf{n}}\right)^2\,ds\end{aligned} \tag{10}$$

combining (9) and (10), we have proved the following inequality:

Corollary 3.5 *Let $v \in \mathcal{H}(\Omega) \cap C^1(\bar{\Omega})$, where $\Omega = B(0,1)$ in $\mathbb{R}^{\mathrm{N}}$ ($\mathrm{N} > 2$). Then,*

$$\frac{1}{\mathrm{N}-2}\int_{\partial\Omega}\left(|\nabla_s v|^2 + (N-3)\left|\frac{\partial v}{\partial \mathbf{n}}\right|^2\right)\,ds \leq \int_{\partial\Omega}\left(v + \frac{\partial v}{\partial \mathbf{n}}\right)^2\,ds\,. \quad \square \tag{11}$$

If $\mathrm{N} = 3$ and v is a solution of (7), then from (11) we get

$$\int_{\partial\Omega} |\nabla_s v|^2\,ds \leq \int_{\partial\Omega} (g^2 - |\nabla_s v|^2)_+\,ds\,.$$

Then

$$\int_{A_-} |\nabla_s v|^2\,ds + \int_{A_+} |\nabla_s v|^2\,ds \leq \int_{A_-} (g^2 - |\nabla_s v|^2)\,ds\,, \tag{12}$$

and hence

$$2\int_{A_-} |\nabla_s v|^2\,ds + \int_{A_+} |\nabla_s v|^2\,ds \leq \int_{A_-} g^2\,ds\,.$$

Since

$$\int_{A_+} |\nabla_s v|^2 \, ds > \int_{A_+} g^2 \, ds ,$$

then we can state the following

Proposition 3.6 *Let $g \in C_+(\partial\Omega)$ and let v be a classical solution (assumed to exist) of (7). Then*

$$\int_{A_-} g^2 \, ds > \int_{A_+} g^2 \, ds , \tag{13}$$

and, in particular $\mathrm{meas}(A_-) > 0$. *Moreover*

$$\mathrm{meas}\left(\left\{x \in \partial\Omega : |\nabla_s v| \leq \frac{g}{\sqrt{2}}\right\}\right) > 0 , \tag{14}$$

and

$$\int_{A_+} |\nabla_s v|^2 \, ds < \|g\|^2_{L^2(\partial\Omega)} . \tag{15}$$

(Here $\mathrm{meas}(C)$ *denotes the surface-area measure of a set $C \subseteq \partial\Omega$.)*

Proof. That $\mathrm{meas}(A_-) > 0$ comes from (13). To prove (14) we use the descomposition $A_- = B_1 \cup B_2$ where

$$B_1 = \left\{x \in \partial\Omega : \frac{g}{\sqrt{2}} < |\nabla_s v|\right\}$$

and

$$B_2 = \left\{x \in \partial\Omega : \frac{g}{\sqrt{2}} \geq |\nabla_s v|\right\} .$$

From inequality (12) we deduce that

$$\begin{aligned} 0 &\leq \int_{A_+} |\nabla_s v|^2 \, ds \leq \int_{A_-} (g^2 - 2|\nabla_s v|^2) \, ds \\ &= \int_{B_1} (g^2 - 2|\nabla_s v|^2) \, ds + \int_{B_2} (g^2 - 2|\nabla_s v|^2) \, ds . \end{aligned} \tag{16}$$

Calling $f(x) = g^2(x) - 2|\nabla_s v(x)|^2$ for $x \in \partial\Omega$, it is obvious that $f(x) < 0$ on B_1, whereas $f(x) \geq 0$ on B_2. Then, if $\mathrm{meas}(B_2) = 0$ we arrive to a contradiction since $\mathrm{meas}(B_1 \cup B_2) = \mathrm{meas}(A_-) > 0$. Inequality (15) immediately follows from (12). The proof is complete. □

Remark 3.4 If $g > 0$, (14) also follows from Remark 3.3. □

Although we have not proved that $A_+ = \emptyset$, the Proposition 3.6 can be considered as a partial result in this direction.

About the existence of solutions of (7), it should be noted that in contrast to the problem (3), the problem (7) automatically is oblique using the terminology followed in [18]. In fact, for a general formulation

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ G(x, u, \nabla u) = 0 & \text{on } \partial\Omega, \end{cases} \tag{17}$$

the problem (17) is *oblique* if, at $\Gamma = \partial\Omega \times \mathbb{R} \times \mathbb{R}^3$, the following inequality is satisfied:

$$\chi = \langle G_{\mathbf{p}}, \mathbf{n} \rangle > 0\,, \tag{18}$$

where $G_{\mathbf{p}}$ denotes the (weak) partial derivative with respect to $\mathbf{p}$ when G is expressed in dummy variables $(x, z, \mathbf{p}) \in \Gamma$. Note that in the case of the original Backus problem (1) $G(x, z, \mathbf{p}) = |\mathbf{p}|$ and then G is oblique if and only if $\langle \mathbf{p}, \mathbf{n} \rangle > 0$ (*i.e.* the condition depends on $\partial u/\partial \mathbf{n}$ which is a priori unknown; the same can be said for problem (3)).

Lemma 3.7 *Problem* (7) *is oblique.*

Proof. In these variables the boundary operator in (7) is given by

$$G(x, z, \mathbf{p}) = z + \langle \mathbf{p}, \mathbf{n} \rangle - \sqrt{(g^2(x) - |\mathbf{p}_t|^2)_+} \tag{19}$$

where $\mathbf{p}_t = \mathbf{p} - \langle \mathbf{p}, \mathbf{n} \rangle \mathbf{n}$ is the tangential projection of $\mathbf{p}$. Differentiating G with respect to $\mathbf{p}$ we get that for any prescribed (x, z)

$$G_{\mathbf{p}} = \begin{cases} \mathbf{n} & \text{if } |\mathbf{p}_t| > g(x), \\ \mathbf{n} + \dfrac{\mathbf{p}_t}{\sqrt{g^2(x) - |\mathbf{p}_t|^2}} & \text{if } |\mathbf{p}_t| < g(x), \end{cases}$$

and this proves that $\chi = 1$. □

Although G given by (19) is not regular enough as to may apply a known existence theorem for oblique nonlinear boundary value problems (see [18]), it seems possible to approach G by more regular functions G_ε and to obtain an existence theorem for (7) by passing to the limit (see [8]).

Remark 3.5 It is interesting to note the following property of the operator (19). Let $\lambda > 0$. Observing that

$$\langle \mathbf{p} + \lambda \mathbf{n}, \mathbf{n} \rangle = \langle \mathbf{p}, \mathbf{n} \rangle + \lambda ,$$

and since the tangential projections of $\mathbf{p}$ and of $\mathbf{p} + \lambda \mathbf{n}$ coincide, then we have

$$G(x, z, \mathbf{p} + \lambda \mathbf{n}) - G(x, z, \mathbf{p}) = \lambda , \tag{20}$$

for all $(x, z, \mathbf{p}) \in \Gamma$. From (20) we can conclude that the function $G(x, z, \mathbf{p})$ is strictly increasing with respect to $\mathbf{p}$ in the normal direction to $\partial\Omega$ at x. G.Barles ([5]) has recently proved that non-linear boundary value problems with this property have, under some other additional conditions, a unique *viscosity solution* (see [5, §I]) in $C(\bar{\Omega})$. $\square$

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