Marine Geodesy and its future

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SUMMARY

Marine geodesy (MG) is the application of geodetic tasks to the parts of the earth's surface covered by oceans. Since they amount to two-thirds of the entire surface MG is of high importance both to theoretical and practical geodesy. MG makes use of cognitions and procedures of continental geodesy (CG) but is distinguished from the latter by some essential characteristics.

The practical importance of MG follows from an increasing importance of the oceans as an economic domain and as a reserve of foodstuffs and raw materials for the world. About one-fifth of the crude oil demand of the world is already being extracted from the shelf region in the earth crust; by the year 2000 it will be about one half. On the bottom of the seas are large quantities of manganese, copper, nickel, cobald and titanium that exceed the known deposits on continents. The latter are constantly diminishing by exploitation whereas the former are currently increasing by sedimentation. They are added by deposits in the crust below the seas the exploitation of which is already technically feasible.

The large quantities of fish and plants living in the seas are already contributing to nutrition of man. Such contribution will greatly increase with the establishment of fish farms in the sea and with the use of other procedures to obtain protein from animals and plants of the sea. Thus the importance of the sea as a foodstuff reserve for the steadily multiplying mankind will incrase too. Modern technologies already now permit the erection of industries in the sea, the utilization of the sea as a living space for man, the creation of centres for the defence and for the control of the outer space of earth. In the future this will be possible on a larger scale.

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From the statements made hitherto follows both an increase in the economic value of the sea floor (that in some cases is approaching the ground value on land and may exceed it in the future) and the importance of the sea itself as a foodstuff reserve and a living space for mankind.

The same applies to MG providing the prerequisites for an economic exploitation of the oceans. Thus the development of MG is an important task of modern geodesy.

In the paper the problems of MG are discussed. After the description of the observation data available with MG the determination of Controlpoints on the floor and on the surface of the sea are investigated. Thereon the topographic survey of the sea floor, the constructions of topographic maps and methods for the determination of the marine geoid and the seasurface topography are discussed. Finally an outlook on the future development of MG is given. From this the following future tasks of MG are predicted.

Position determination on the sea, in the sea and on the sea floor, production of topographic and thematic charts of the sea bottom, determination of the sea surface topography and of the sea geoid, determination of the geodynamic variations of the earth's crust, of the geoid and of the sea topography, establishment of test areas for testing instruments and techniques, conclusion of international agreements for carrying out experiments and practical tasks. Solutions already existing and solution sets are to be further developed to increase their accuracy, complete their statement and reduce the time required therefore.

1. TASKS OF MARINE GEODESY

Geodesy has to fulfil a geometric and a physical task. The geometric task is to determine the shape and position of the earth's surface and of natural and artificial objects situated thereon as well as to describe and represent them by means of coordinates and maps. (This task also includes engineering surveying in conjunction with technical projects, laying out the project, and determining changes of territory and objects.) The physical task aims at determining the structure of the field of gravity on the earth's surface and in outer space. Both these tasks are to be carried out as completely, accurately and quickly as possible so that changes taking place and being expected can be distinguished as a function of time. The final aim should be a real-time reporting on the geometric behaviour and the structure of the field of gravity on and outside the earth.

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On the continents the data can be determined on the surface to be measured and repeated any time. On the ocean floor this is hardly possible. Measurements

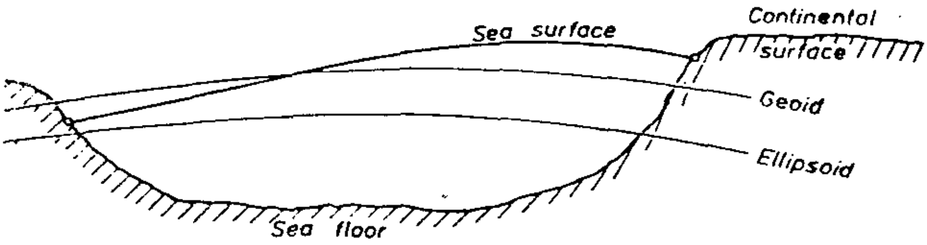

Figure 1

have to be carried out on the constantly moving surface of the sea. Repeats give new data determining adjacent but different configurations for the unknown parameters. MG has to cover more reference surfaces than CG: the surface of the earth crust below the oceans (sea floor), the sea surface assumed to be undisturbed, and the geoid as a special potential surface of the earth body (see *Fig. 1*).

There are also some differences in the type and determination of data. CG data are obtained in the atmosphere whereas in MG an essential portion of the measurements have to be made through the sea. So the restriction of measuring means is greater, their accuracy is lower. Refraction of the atmosphere is added by refraction of the sea.

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The same applies to MG providing the prerequisites for an economic exploitation of the oceans. Thus the development of MG is an important task of modern geodesy. Its scientific bases will have to be investigated systematically and the development of its procedures and instruments carried on with an intensity similar to that used in CG. Since the sea is doubtlessly of higher importance for the further development of mankind than the moon and the planets of the solar system it may be demanded to make available similar amounts of money for the development of MG as for extraterrestrial geodesy.

The general aims of MG described lead to the formulation of actual geodetic tasks. Having regard to the increasing value of the ocean floor it will be necessary to have a sea lot register to establish, secure and reproduce at any time the boundaries between exploitation areas on the sea floor and on the sea surface. As a basis for such register it will be necessary to establish a system of control points from which the boundaries can be determined by means of measuring data. These MG control points may be, as on continents, permanently marked points on the sea floor whose space position can be fixed with high accuracy, or a sufficiently dense system of geodetic satellites with known ephemerides. Topographic maps of the sea floor will be required for technical-economic projects. They should contain a complete and accurate representation of the sea floor and the navigation systems installed, just as the nautical charts used hitherto. These are navigation instruments; topographic maps should additionally contain the data required for the geophysical exploration and the realization of engineering projects.

With the aid of MG control points and maps it will be possible to divide the sea floor and the sea surface and to substantiate legal rights of property, prospecting licences and servitudes. Also the position of objects on the sea, in the sea and on the sea floor can be ascertained. These points will also help to find lost objects (ships, aircraft, satellites) on the surface, in the sea and on the sea bottom. They may also be used to record natural phenomena and approaching catastrophes, and to establish warning systems (for Tsunamis).

The knowledge of the geoid, the sea surface topography, the sea floor and the structure of the gravity field on the sea, in the sea and on the sea floor will be of importance for scientific and practical problems, likewise the knowledge of recent changes of these data in position and height. MG (like CG) will have to elaborate a repeatedly controlled and statistically illustrated description of phenomena observed but not their interpretation or explanation. This will provide important bases for the handling of problems arising in oceanography and other geosciences in connection with the sea.

Table 1 gives a systematic survey of MG tasks and their practical and scientific applications (substantially taken from /1/). The basic geometric task is supposed to be the determination of control points and position points derived from the latter, the basic physical task the determination of a gravity model of earth. The solution of these tasks is a previous condition for the subsequent tasks of mapping, technical investigation, science, geophysical exploration and oceanography. Test fields will have to be established for testing and calibrating instruments and procedures.

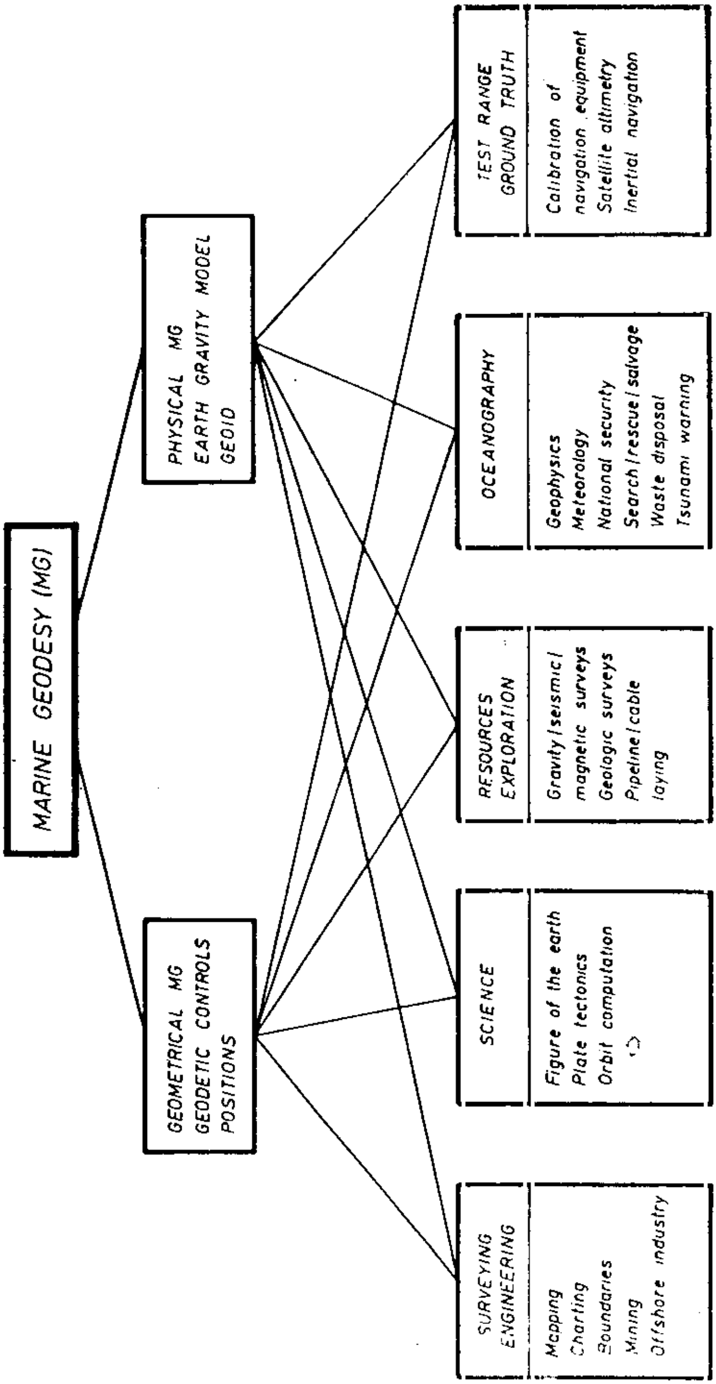

MARINE GEODESY (MG)
PHYSICAL MG
EARTH GRAVITY MODEL
GEOID
GEOMETRICAL MG
GEODETIC CONTROLS
POSITIONS
SURVEYING
ENGINEERING
Mapping
Charting
Boundaries
Mining
Offshore industry
SCIENCE
Figure of the earth
Plate tectonics
Orbit computation
RESOURCES
EXPLORATION
Gravity/seismic/
magnetic surveys
Geologic surveys
Pipeline/cable
laying
OCEANOGRAPHY
Geophysics
Meteorology
National security
Search/rescue/salvage
Waste disposal
Tsunami warning
TEST RANGE
GROUND TRUTH
Calibration of
navigation equipment
Satellite altimetry
Inertial navigation

Table 1

POSITIONAL ACCURACY REQUIREMENTS

GEODETIC OPERATIONS	Precision in m			Accuracy in m		
	± N	± E	± H	± N	± E	± H
Control points	1	1	1	10	10	5
Test nets stations	1	1	0.3	10	10	5
Gravity base stations	10	10	1	10	10	5
Geoid	—	—	0.1	—	—	0.5
OCEAN PHYSICS						
Mean sea level	—	—	—	50-100	50-100	0.1
Stationary buoys	10	10	—	10	10	—
Drifting buoys	50-100	50-100	—	50-100	50-100	—
Sea floor spreading	0.1	0.1	0.1	—	—	—
Ice sheet motion	1-5	1-5	—	—	—	—
SEARCH, RESCUE, SALVAGE	1-10	1-10	—	20-100	20-100	—
OCEAN RESOURCES						
Geophysical surveys	10-100	10-100	5	—	—	—
Drilling	1-5	1-5	1-5	—	—	—
Pipeline, cable laying	1-10	1-10	—	—	—	—
Dredging	2-10	2-10	—	—	—	—
TRACKING STATIONS	—	—	—	10	10	—

N: North E: East H: Height

Table 2

Of importance are the demands of accuracy made at present by the various disciplines interested in the results of MG. They are contained in *Table 2* (as established on the basis of enquiries made, first communicated in /1/; «precisión» denoting the mean accuracy of the procedure and «accuracy» the mean position accuracy in the coordinate system chosen). Remarkable are the high demands made on geoid altitudes and on the horizontal position for recent motions of ± 0.1 m as well as the accuracy of ± 1 m for the coordinates of control points. Similar demands of accuracy are being made on MG as on CG although measuring conditions are much more difficult in MG.

2. OBSERVATION DATA

In CG, measurements are (as a rule) carried out from earth fixed stations; also the target points are (as a rule) fixed. Thus it is possible to repeat individual measurements and increase their accuracy. In MG, however, most of the stations and target points are (as a rule) situated on moving objects (ships, platforms, satellites) and only few in fixed places (coast, sea floor). The data of measurement belonging to a configuration must therefore be determined simultaneously, an increase in accuracy is possible by simultaneous repetition of all measurements in adjacent configurations. Simultaneous measurements can only be dispensed with if the path curves of moving stations or target points are known with such precision that centering can be carried out. In MG, tasks with time-dependent data of measurement are considered as a rule that in CG are termed dynamic.

Is MG, only part of the measurements are made on or outside the sea, i.e. in the atmosphere, most of them are carried out in the sea. On principle, all dynamic measuring techniques can be performed on and outside the sea: distances (light and micro waves), Doppler data, directions (by photographic extension of the astronomic system), space angles and gravimetric data. In most cases it will be necessary to stabilize the measuring device. Only theodolite measurements cannot be made at present.

In the sea, the techniques used for direction and distance measurements on land cannot be applied because of the poor propagation conditions of light and micro waves. Light and micro waves are therefore replaced by ultrasonar (US). Since US is moving in the sea along a path curve to be determined by density measurement at determinable velocity the space distance between the measuring points can be ascertained by integration from the running time measured for a US signal. A density profile has to be established in addition to the running time. As a rule, the measurement is made from objects on the surface or in the sea to fixed points on the sea floor. Reflectors are installed at these points which on receiving an (acoustic) signal emit a US signal of certain frequency. Beside these devices, called transponders, there are beacons and pingers which emit signals constantly or at intervals, or are induced to do so by electric signals coming via cables.

Measurement of the US distance can (in case of fixed end points) be repeated any time. It is also possible to simultaneously measure distances from one point to a number of points and to determine distance differences by means of acoustic Doppler techniques. Quotients of distances follow from running-time quotients if the velocity distributions are equal for both distances. In this case it is possible to do without the expensive measurement of density profiles.

By measuring adjacent distances it is possible to determine the direction of the measuring beam (like in the Minitrac technique) but the accuracy obtained is still very poor.

Gravity data on the sea surface can be measured with sea gravimeters. On account of constant vertical and horizontal accelerations their precision is about 100 times worse than on land, in spite of stabilization of the platform. Gravity measurements on the ocean floor provide more accurate values but they are difficult and expensive. Also measurements in submarines are possible; on account of gravity dependence on latitude they are mostly used for navigation.

Summarizing the following can be said: In MG, all measuring instruments and techniques can be used to determine points in, on and above the sea that have been developed for dynamic CG. In the sea, only US measurements of the dynamic version are possible. Hence it follows that all MG techniques are dynamic techniques in which the time of measurement has to be considered as another coordinate or eliminated by the demand for simultaneousness.

The accuracy of gravity measurements on or in the sea is limited, the physical techniques on and in the sea are therefore not of the same importance for MG as for CG.

3. DETERMINATION OF CONTROL POINTS ON THE SEA FLOOR

MG control points are transponders installed on the sea floor the space position of which is defined in a unified geodetic datum. From such MG control points it is possible to derive the space position of points on and in the sea by means of US measurements. They are starting points for detailed surveying and the coordination of objects on and in the sea. They may be used to control navigation techniques and permit the finding of lost objects in the sea. Therefore the establishment of a sufficiently dense system of control points is to be regarded as an important task of MG.

Since position determination with US distances is only possible if distances to at least three known points are available MG control points are arranged in groups of at least three points (transponders).

On principle, the position of control points could be determined by US trilateration in the sea. For practical reasons this method is, as a rule, not used. Methods are being used that establish, with the aid of measurements on the sea, the connection between known points (coast, satellite) to the control points on the sea floor. From (surveying) ships, US distances are measured to the control points (on the sea floor) and, at the same time, electromagnetic (EM) distances, directions, Doppler data (distance differences) to the starting points (coast, satellite).

MG control points may be defined by two methods. In the first method, precise local coordinates for the MG points are first determined by local US measurements and fixed in their relative position. By means of measuring data to known starting points the elements of a linear transformation are then determined by means of which the local coordinates are transferred into the final geodetic datum. In the second method, the local measurements are only used to determine approximate coordinates for the MG control points; the final coordinates for the control points are defined in one founding by means of adjustment. Subsequently, the theory of the two methods is described at first, and then the determination of the local coordinates and of the orientation of the local system is being dealt with.

3.1. General solution

S be the measuring place on the ship, C_i the $i \geqslant 3$ control points to be determined (measuring centres of transponders) on the sea floor; P be the starting point with the known coordinates **x** (on the coast or in the satellite). In S, the US distances s_i to the points C_i and the EM distance s to P are assumed to be measured at the same time *(Fig. 2)*. Besides it is assumed that in special cases instead of or in addition to s also the direction $\mathbf{r} = (PS)$ has been determined by photographic extension of the stellar sky. Also the case where distance differences Δs from S to points P, $\bar{P}$ (with Doppler instruments) are determined is to be included.

The task is now to determine the coordinates $\mathbf{x}_i$ for the MG control points from the known coordinates **x** of a number of points P and the respective j groups of simultaneous measurements (s_{ij}, s_j) or $(s_{ij}, \mathbf{r}_j)$ or $(s_{ij}, \Delta s_j)$.

Each measuring group gives rise to conditional equations of the form:

$$
\begin{aligned}
s_i^2 &= (\mathbf{x}_s - \mathbf{x}_i)^2, \; i = 3 \\
s^2 &= (\mathbf{x} - \mathbf{x}_s)^2 \\
\Delta s &= \bar{s} - s \\
\mathbf{r} &= \frac{1}{s} (\mathbf{x} - \mathbf{x}_s), \; \mathbf{r}^2 = 1
\end{aligned}
\qquad (1)
$$

One of the measuring groups (s_{ij}, s_j), $(s_{ij}, \Delta s_j)$ provides $n = (i+1)$ conditional equations for the $u = 3(i+1)$ unknown coordinates of S_j and C_i. If the measurement

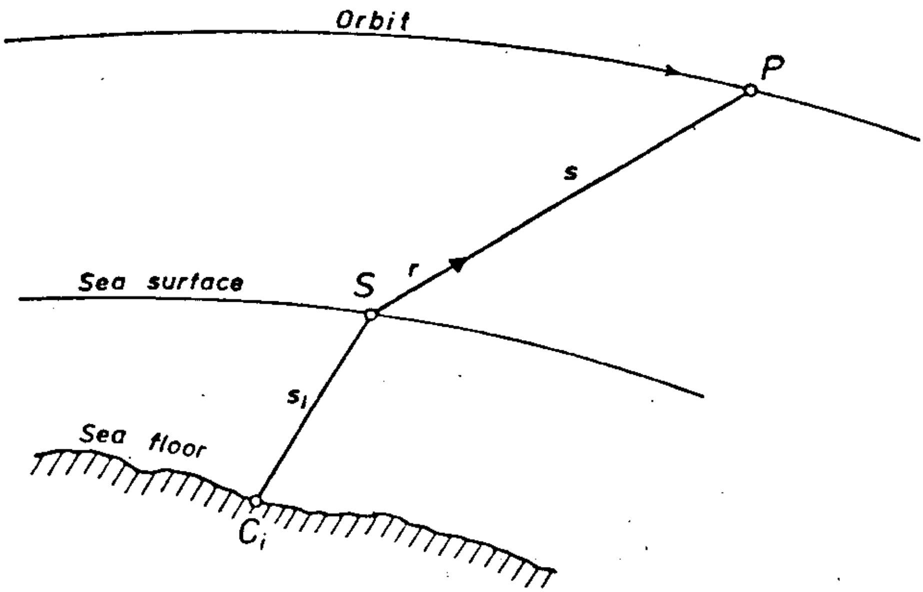


Figure 2

is made j times there will be a total of $n_j = j(i+1)$ equations for $u_j = 3(i+j)$ unknown coordinates of C_i and S_j. Determination is possible if $n_j \geqslant u_j$. Hence follows a relation between the number i of control points and the number j of measuring groups required for their determination.

$$j \geqslant \frac{3i}{i-2} \tag{2a}$$

Thus, j=9 measuring groups are required for i=3, for i > 3 the number j of necessary measuring groups decreases.
A measuring group (s_{ij}, $\mathbf{r}_j$) provides n = (i+2) equations for u=3(i+j) unknowns. Therefore, for j measuring groups the following relation holds:

$$j \geqslant \frac{3i}{i-1} \tag{2b}$$

In this case, at least 5 measuring groups are required for determining i=3 points C_i. Since in the Eqs. (1) the unknowns $\mathbf{x}_i$ and $\mathbf{x}_s$ are present in nonlinear form we shall have to start from approximations $\mathbf{x}_i'$ and $\mathbf{x}_s'$. They can be determined by applying navigation techniques. From three known ship places S_j follow approximate values $\mathbf{x}_i'$ for the points C_i by measuring the US distances. The ship place S_j belonging to a measuring group can again be determined from the known apoproximations $\mathbf{x}_i'$.

The formulas for computing the coordinates of a point from three measured distances are given in *Annex 1*. The approximations obtained are conveniently corrected by adjustment according to variation of coordinates. If v denotes the correction of the measured value m, m′ the value following from the approximations $\mathbf{x}'$ according to Eqs. (1), and dm the variation caused by $d\mathbf{x}$, $d\mathbf{x}_s$ the following equation will hold:

$$\begin{aligned} m + v &= m' + dm \text{ or} \\ v &= dm + \ell \\ \ell &= m' - m \end{aligned} \tag{3}$$

For direction coordinates

t hour angle
δ declination

Eq. (3) is a vector equation corresponding to two scalar equations (see *Annex 2*). The combination of all equations gives the system

$$\mathbf{v} = \mathbf{A}_1 d\mathbf{x}_s + \mathbf{A}_2 d\mathbf{x}_i + \mathbf{l} \tag{4}$$

for the unknowns $d\mathbf{x}_s$ and $d\mathbf{x}_i$ of all ship and control points. Observing the weight matrix **P** and the adjusting requirement $\mathbf{v}^T \mathbf{P} \mathbf{v} = \min.$, a system of normal equations follows for the determination of $d\mathbf{x}_s$ and $d\mathbf{x}_i$ of the 3(i+j) unknowns.

$$\begin{bmatrix} A_1^T PA_1 & A_1^T PA_2 \\ A_2^T PA_1 & A_2^T PA_2 \end{bmatrix} \begin{bmatrix} dx_s \\ dx_i \end{bmatrix} + \begin{bmatrix} A_1^T Pl \\ A_2^T Pl \end{bmatrix} = 0 \tag{5a}$$

Herewith, corrected values

$$x_s'' = x_s' + dx_s \, , \quad x_i'' = x'_i + dx_i \tag{5b}$$

can be computed. The iterative application gives final coordinates x_s, x_i, their errors and correlations.
Since only the coordinates x_i of the control points C_i are needed the corrections dx_s of the ship positions can be eliminated either in the course of numerical computation or by analytical methods.

The analytical elimination reduces the number of unknowns and is therefore of practical use. In the technique described in the following it is first assumed that i=3 control points are existing and the system (4) contains only three conditional equations for dx_s.

$$r_i^T dx_s = r_i^T dx_i - \ell_{si} + v_{si}$$

From the latter, dx_s can be represented explicitly with the aid of the reciprocal vectors r_i^* of r_i (see *Annex 3*).
l and v be the vectors of the absolute terms and corrections represented in the system r_i^*

$$l = \sum_{i=1}^{3} \ell_i r_i^* \, , \; v = \sum_{i=1}^{3} v_i r_i^* \tag{6a}$$

W a matrix formed with (r_i^*, r_i)

$$W = (W_1, W_2, W_3) \, , \; W_i = (r_i^* r_i^T) \tag{6b}$$

and finally

$$dx_T = (dx_1^T \, , \, dx_2^T \, , \, dx_3^T) \tag{6c}$$

the vector of all corrections dx_i. With them follows (according to Eq. (6b), *Annex 3*) the explicit relation:

$$dx_s = W \, dx + l + v \tag{6d}$$

The latter can be simplified if the mutual position of the control points C_i is known from local measurements (line crossing, cloverleaf maneuver). In this case, only the parameters of the linear transformation have to be determined by which the local system is transformed into the geodetic datum of the starting points (originates). Adjacent systems, i.e. differential parameters (shift dc, scaling $d\mu$ and rotations **da**), are always existing. According to the derivations listed in *Annex 4* the following equation of transformation holds:

$$\mathbf{dx}_i = \mathbf{dc} + d\mu\, \mathbf{x}_i + \mathbf{X}_i\, \mathbf{da} \tag{7a}$$

Substituting this into Eq. (6d) the following will hold with $\Sigma\, \mathbf{r}_i^* \mathbf{r}_i^T = \mathbf{E}$

$$\mathbf{dx}_s = \mathbf{dc} + d\mu\, \mathbf{Wx} + \mathbf{WXda} + \mathbf{l} + \mathbf{v} \tag{7b}$$

where $\mathbf{x}$, $\mathbf{X}$ are quantities defined by the vectors $\mathbf{x}_i$.

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \end{bmatrix}, \quad \mathbf{x}_i = \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{(i)}, \quad \mathbf{X} = \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \\ \mathbf{X}_3 \end{bmatrix}, \quad \mathbf{X}_i = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix}_{(i)} \tag{7c}$$

In most cases, the scale of the US distances is known and thus $d\mu = 0$; in some cases, also the orientation (from the local measurement), then also $\mathbf{da} = \mathbf{0}$. Unknown remains the shift **dc**, and from Eq. (7b) follows the simple relation:

$$\mathbf{dx}_s = \mathbf{dc} + \mathbf{l} + \mathbf{v} \tag{7d}$$

Substituting the Eqs. (6d) or (7b,d) into Eq. (4), a system of correction equations follows of the form:

$$\mathbf{Av} + \mathbf{Bz} + \mathbf{w} = \mathbf{0}, \tag{8a}$$

where **v** is the vector of the corrections and **z** the vector of the nine coordinate variations $\mathbf{dx}_i$ and of the seven parameters (**dc**, $d\mu$, **da**) of the transformation. **A**, **B** are fully occupied matrices dependent on the point coordinates and the observation data, **w** is the vector of the absolute terms. This is added by the weight matrix **P** of the observation data. The system (8a) corresponds to the general case of adjustment calculation in which each observational equation contains several corrections and the unknowns. Therefrom follows, according to the requirement $\mathbf{v}^T\mathbf{Pv} = \min.$, the system of normal equations:

$$\mathbf{B}^T (\mathbf{AP}^{-1}\mathbf{A}^T)^{-1}\mathbf{Bz} + \mathbf{B}^T (\mathbf{AP}^{-1}\mathbf{A}^T)^{-1}\mathbf{w} = \mathbf{0} \tag{8b}$$

By eliminating the $\mathbf{dx}_s$, the adjustment problem by variation of coordinates goes over, according to Eq. (5a), with (9+3j) unknowns into a general adjustment problem with unknowns.

If more than i=3 MG control points are to be determined at the same time Eq. (6d) or (7b,d) must also be substituted into the remaining (i-3) equations (4). Hence follows a system of observational equations for the (ij) US distances in which all corrections of 3i unknowns $\mathbf{dx}_i$ occur in each equation.

$$\mathbf{A'v} + \mathbf{B'x} + \mathbf{w'} = \mathbf{0} \tag{8c}$$

The combination with Eq. (8a) gives a general adjustment problem with (3i+7) unknowns the solution of which is described by equations in the form of Eq. (8b).

3.2. **Determination of local coordinates**

Local coordinates for the MG control points C_i can be determined by US and EM trilateration from various positions S_j of a survey ship. From j ship places, i US distances to C_i, i.e. a total of n=ij US distances, are measured at the same time. Since $u = 3(i+j)-6$ coordinates are to be determined and $n \geqslant u$, the following relation holds:

$$j \geqslant 3\,\frac{i-2}{i-3} \tag{9a}$$

Thus, the fixation of i=4 control points is possible by simultaneously measuring four quadruples each of distances s_{ij} to these points j=6 ship places. The determination becomes, however, uncertain (critical case) if the points C_i lie in one plane. Since this will often occur in an approximate way the technique can be only employed to a limited extent.

The uncertainty will be eliminated by additionally determining k distances between the ship places S_j. In this case, the counting of the conditional equation and unknowns gives the inequation (see /2/):

$$j \geqslant \frac{3i-k-6}{i-3} \tag{9b}$$

Since this is satisfied for i=4, j=4, j=(6-k), the coordination of four control points C_i of j=4 ship places, for instance, is possible by measuring k=2 EM distances between the S_j along with the quadruples of the US distances. This is simply done by using two survey ships measuring simultaneously the US distances to the C_i and their mutual EM distance. From two such measuring groups follows the determination of the four MG control points. Any further measuring group yields redundant information.

For i=4, k=3, Eq. (9b) gives j=3. If three survey ships are available simultaneously measuring quadruples of US distances to the four MG points and the EM distances between their points of measurement, the local position of the MG points will be determined by one such measuring group; one more will give redundancy in determination.

If the topography of the sea surface is known, only the horizontal coordinates of the points S_j and all three coordinates of the points C_i will be unknown. Besides, only the horizontal shifts and the rotation around the z-axis, i.e. three parameters, cannot be determined. n = (ij+k) observation data are faced with u = (3i+2j−3) unknowns. Hence follows the condition

$$j \geqslant \frac{3i - k - 3}{i - 2} \tag{9c}$$

which, for i=3, has the solution j ⩾ (6-k). In this case, the space position of three MG control points can be defined by six triples of simultaneous US distances or by four such triples and two EM distances between points S_j.

Any range determination provides a conditional equation of the type of Eq. (*Annex* 2, 1b). Since for the local position six coordinates can be chosen freely, the altitudes of the ship places S_j are chosen equal to the geoid altitudes (or equal to zero), and the position of a point and the orientation of a side are determined by navigation. The observational equations have the form:

$$\begin{aligned} v_{ij} &= \mathbf{r}_{i\,j}^{T}\, d\mathbf{x}_j - \mathbf{r}_{ij}^{T}\, d\mathbf{x}_i + \ell_{ij} \\ l_{ij} &= s'_{ij} - s_{ij} \end{aligned} \tag{10a}$$

The combination gives the system:

$$\mathbf{v} = \mathbf{Az} + \mathbf{l}, \text{ weight matrix } \mathbf{P}$$

Hence follow the normal equations

$$\mathbf{A}^T\mathbf{PAz} + \mathbf{A}^T\mathbf{Pl} = \mathbf{0}, \tag{10b}$$

from which the coordinate corrections, their errors and correlations can be computed.

A special case exists if the US velocity is not known but may be assumed to be constant in the operational area.

If s'' denote the distances computed with approximate velocity the correct distances s can be obtained therefrom with the aid of a scale factor μ.

$$\mu s'' = s \;,\; \mu ds'' = ds \;,\; \mu v''_s = v_s \tag{11a}$$

From the relation according to Eq. (1)

$$(\mu s'')^2 = (\mathbf{x}_s - \mathbf{x})^2 \tag{11b}$$

follows the observational equation:

$$\begin{aligned} v_{si} &= \mathbf{r}^T(d\mathbf{x}_s - d\mathbf{x}_i) - s_i \frac{d\mu}{\mu} - l_{si} \\ l_{si} &= s'_i - s_i \end{aligned} \tag{11c}$$

The unknown $d\mu$ can be eliminated therefrom with the aid of the equation for a distance s_j. The resulting equation

$$s_j v_{si} - s_i v_{sj} - (s_j \mathbf{r}_i^T - s_i \mathbf{r}_j^T) d\mathbf{x}_s + s_j \mathbf{r}_i^T d\mathbf{x}_i - s \mathbf{r}_j^T d\mathbf{x}_j + w = 0$$
$$w = s_j l_{si} - s_i l_{sj} \qquad (11d)$$

corresponds to a more general case of adjustment where every observation equation contains several corrections and unknowns. The same result is obtained by eliminating the scale factor from the equations (11b) and starting from the relation:

$$\frac{s_i^2}{s_j^2} = \frac{(\mathbf{x}_s - \mathbf{x}_i)^2}{(\mathbf{x}_s - \mathbf{x}_j)^2} \qquad (11e)$$

The geometric significance of this relation as defined by distance relations has been thoroughly dealt with in /3/. In this case, the counting of the conditional equations and unkinowns gives:

$$j \geqslant \frac{3i - k - 5}{i - 3} \qquad (11f)$$

The position of four MG control points can therefore be determined for k=0 (one survey ship) by j=7, and for k=1 and k=2 (two survey ships) with j=6 and j=5 quadruples of US distances, respectively.
For the case of known z_j −values the relation

$$j \geqslant \frac{3i - k - 2}{i - 2} \qquad (11g)$$

holds which for i=3 has the solution $j \geqslant (7\text{-}k)$. The determination of three MG control points is therefore possible from seven triples of simultaneous US distances. The relative position of MG control points can be determined also by measuring their horizontal distances and depths by line crossing and cloverleaf methods. In the former, the survey ship crosses the side to be determined (C_i C_k) on an approximately rectangular course, and measures in points S_j at the time t_j simultaneously the distances s_{ji} and s_{jk} to C_i, C_k. The minimum of the distance sum

$$(s_{ji} + s_{jk}) = F(t_j) \qquad (12a)$$

corresponds to the values (s_{mi}, s_{mk}, t_m) measured in the normal plane through (C_i C_k) in the point of intersection M (see *Annex 5*). With known depths z_M, z_i, z_k, the horizontal or space distance between C_i and C_k can be computed therefrom.

$$s_{ik} = \sqrt{s_{mi}^2 - (z_M - z_i)^2} + \sqrt{s_{mk}^2 - (z_M - z_k)^2} \qquad (12b)$$

For determining three MG points, the crossing courses shown in *Fig. 3a,b* are conveniently used. In the case (a), the points of intersection are determined only once but they fulfill the demand F(t) = min. in all severity. In the case (b), the points of intersection are determined twice; they coincide only approximately with the points following from the minimum demand.

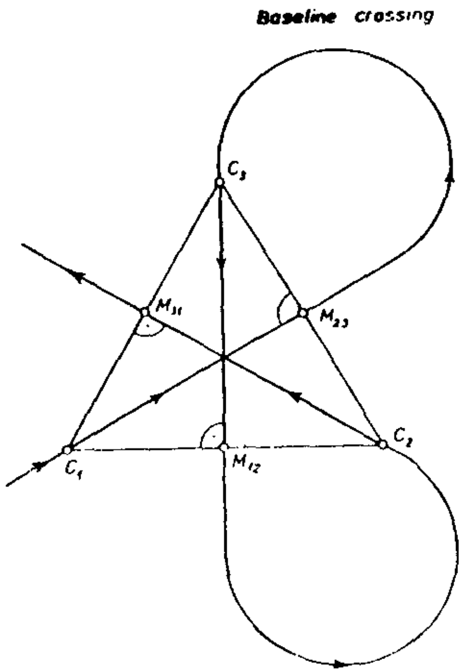


Figure 3a

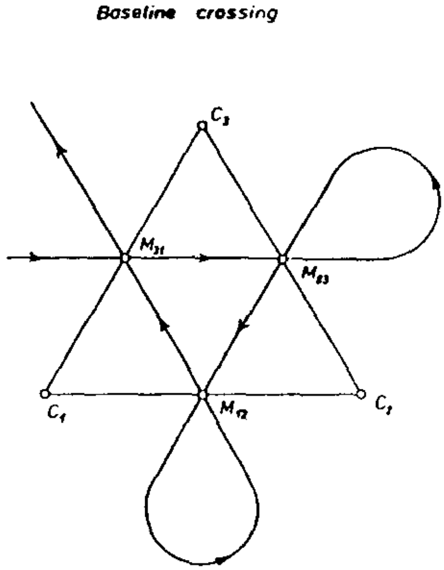


Figure 3b

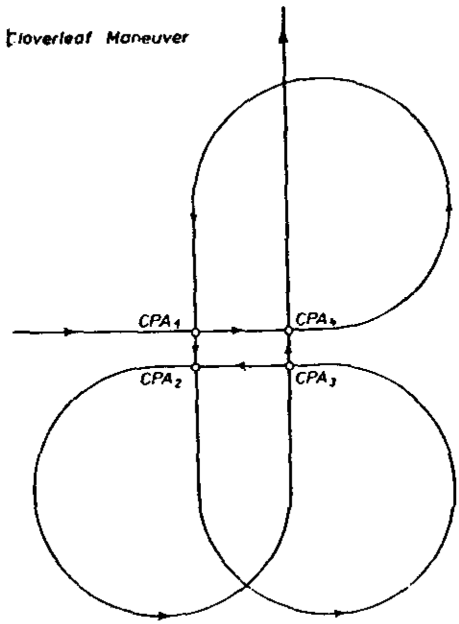


Figure 4a

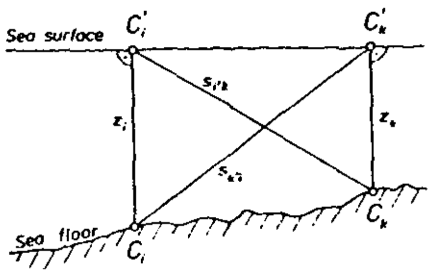


Figure 4b

For determining the depths z_i according to the cloverleaf maneuver, the survey ship steams on a rectilinear course as closely as possible to point C_i. A new course is followed through the computed point of closest approach (CPA) with the smallest distance, and its CPA is determined (see *Fig. 4a*). Finally, the iteration yields the point C_i' situated avove C_i and its depth value. With known sea surface topography, a depth in the geodetic system can be derived therefrom. The same technique is repeated for the end point C_k. A control relation follows from simultaneously measuring the distances $s_{i'k}$ and $s_{k'i}$ from the second end point (see *Fig. 4b*).

$$s_{i'k}^2 - s_{k'i}^2 + z_i^2 - z_k^2 = 0 \tag{12c}$$

By means of the cloverleaf maneuver, the position of the points C_i and the course of the sides (C_i C_k) can be determined by navigation in an approximate way. Thus it becomes possible to choose convenient courses for line crossing.

4. POSITION DETERMINATION ON THE SEA SURFACE

The determination of control points on the sea surface makes only sense if we succed in fixing such points permanently. This is possible, to a limited extent, by means of anchored buoys, platforms and ships. Additional stabilizing equipment is required for the measurement of geodetic data. Control points on the sea surface can be determined with the aid of control points on the shore or in space (satellite), or also with the aid of control points on the sea floor.

4.1. Initial points outside the sea

The space position of points on the sea surface can be determined by the techniques of two- or three-dimensional CG. Since the altitudes are always known, either approximately or —with known sea topography— even precisely, the procedure will generally be simpler than in CG. Particularly suited are the trilateration and accurate navigation techniques with distances, distance differences or quotients because the observation of these data is less disturbed by motion of the observing station than those of directions or angles.

Using a greater number of survey ships, platforms or airplanes it is possible to occupy the nodes of a trilateration net to connect the points on the shore with the new points. Support can be given by measuring individual angles by the sextant method; frequent repetition of the dynamic configuration will result in increased precision. The adjustment gives horizontal coordinates for the new points. The nodes situated between them and the starting points on the shore are different for every configuration due to the ship's motion. The altitudes of the points cannot be determined by such techniques (cf. /4/).

The space position including the altitudes can be determined by the method of geometric or semidynamic satellite triangulation. The satellite, the ephemerides of which are defined by dynamic methods and are therefore known, is regarded as a space target to which distances or distance differences can be measured. The satellite orbit represents a system of control points from which the positions and altitudes of terrestrial points can be determined.

An example therefor is the TRANSIT system used in the USA in which Doppler measurements can be carried out to several satellites circulating along polar orbits. Distance differences Δs from the observing station S to points P, $\bar{P}$ can be determined by integration of the Doppler frequencies. Each measurement Δs defines a rotation hyperboloid with the focal points P, $\bar{P}$ as geometric locus for S. The space position of S in the geodetic datum of the satellite orbit is defined by n=3 such measurements (*Fig. 5*).

In the GEOLE project, distances s are determined in addition to Doppler data Δs (Range-Rate system). The points S are defined in this case by n=2 measuring groups (s, Δs). Each measuring group defines a triangle (P, $\bar{P}$, S) that can be

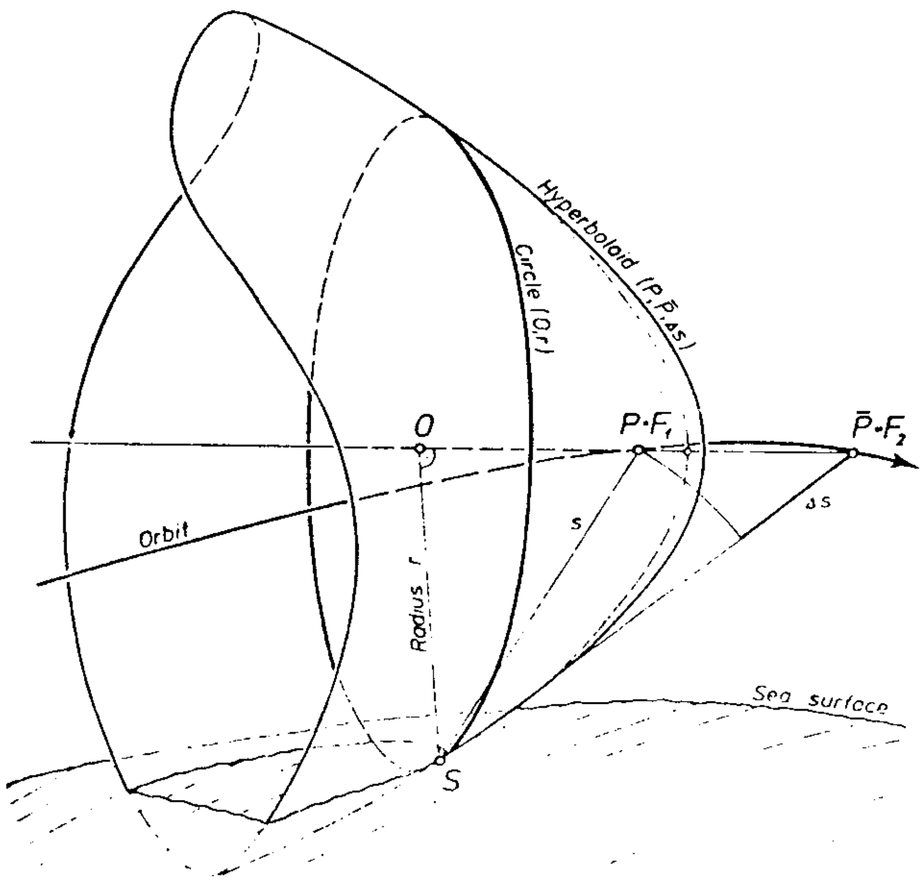


Figure 5

rotated around P$\bar{P}$. Hence follows a circle as the geometric locus for S. Additional n > 2 measuring groups provide redundancy in determination (see /5/).

If the sea topography is assumed to be known, two Doppler or two distance measurements Δs or s or one measuring group (Δs, s) will be sufficient to define a point S. Repeated measurement will increase the accuracy in point determination. Besides, random motion of observing stations is eliminated statistically by a suitable array of measurements with respect to time.

In some cases it will be possible to define the motion of observing stations by (acoustic or inertial) navigation in a local system as a function of time. The orientation of the system will, as a rule, be known from navigation but not its absolute position. In this case, the centering vector **u**(t) required for the reduction of the eccentric measurements on E(t) to S(t=O) can be expressed by local coordinates. Between **u**, the position vector **y** the local system and the position vector **x** in the geodetic datum of the starting points (originates) exists the following relation:

$$\mathbf{u}(t) = \mathbf{T}\,\{\mathbf{y}(t) - \mathbf{y}(o)\} = \mathbf{x}(t) - \mathbf{x}(o) \quad (1)$$

T transformation matrix

The data are centered by geometric relations in the triangle (S,E,P). With the designation according to *Fig. 6* the following vector equation holds:

$$s\mathbf{r} = s_E\mathbf{r}_E + \mathbf{u} \quad (2a)$$

Hence follow equations for computing the data a and **r** reduced on the center S.

$$s = \mathbf{r}^T\mathbf{r}_E s_E + \mathbf{r}^T\mathbf{u} \quad (2b)$$

$$\mathbf{r} = \frac{s_E}{s}\mathbf{r}_E + \frac{1}{s}\mathbf{u} \quad (2c)$$

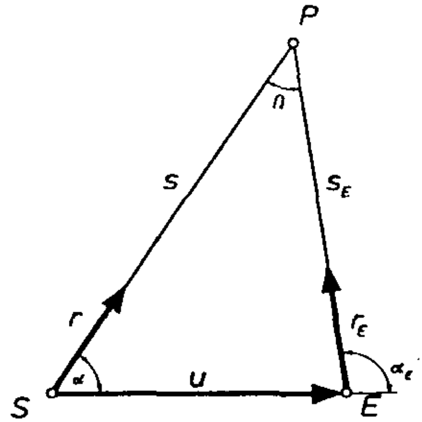


Figure 6

For eccentric Doppler distances $\triangle s = (\bar{s}\text{-}s)_E$ the following holds according to Eq. (2b):

$$\triangle s = \triangle s_E \bar{\mathbf{r}}^T \bar{\mathbf{r}}_E + (\bar{\mathbf{r}} - \mathbf{r})^T \mathbf{u} + (\bar{\mathbf{r}}^T \bar{\mathbf{r}}_E - \mathbf{r}^T \mathbf{r}_E) s_E \tag{2d}$$

In Eqs. (2d,b,c), also the angles α, α_E, β in the triangle (S,E,P) may be introduced. Since, in general, only $\mathbf{u}$, s_E or $_E$ are known the missing data must be derived from the approximate coordinates $\mathbf{x}'$ of S. They are $\mathbf{r}$, $\mathbf{r}_E$ for Eq. (2b), s, s_E for Eq. (wc) and $\mathbf{r}$, $\mathbf{r}_E$, $\bar{\mathbf{r}}_E$, s_E for Eq. (2d).
Centering is also possible if the scale and orientation of the local system $\mathbf{y}$ are known only approximately. In this case, a linear transformation is introduced with the parameters $\mu = (1 + d\mu)$, $\mathbf{R} = \mathbf{E} + d\mathbf{A}$ ($d\mathbf{A}$ = axiator of $d\mathbf{a}$).
Hence follows from Eq. (1) because of

$$\mathbf{x}(t) - \mathbf{x}(o) = \mathbf{R}(\mathbf{y}(t) - \mathbf{y}(o)) = \mathbf{R}\mathbf{u}(t)$$

for the centering vector:

$$\mathbf{x} = \mathbf{u} + d\mu\mathbf{u} + \mathbf{U}d\mathbf{a} \tag{3a}$$

where $\mathbf{u}$ denotes the approximate value following from the approximately oriented system, $d\mu$ the scale correction, $d\mathbf{a}$ the vector of the three rotations, and $\mathbf{U}$ the axiator belonging to $\mathbf{u}$. The equation corresponding to Eq. (2a)

$$s\mathbf{r} = s_E \mathbf{r}_E + \triangle x = s_E \mathbf{r}_E + \mathbf{u} + d\mu\mathbf{u} + \mathbf{U}d\mathbf{a} \tag{3b}$$

also contains the unknowns $d\mu$, $d\mathbf{a}$ so they will also enter into the centered data.

$$s = \mathbf{r}^T \mathbf{r}_E s_E + \mathbf{r}^T \mathbf{u} + \mathbf{r}^T \mathbf{u} d\mu + \mathbf{r}^T \mathbf{U} d\mathbf{a} \tag{3c}$$

$$\mathbf{r} = \frac{s_E}{s} \mathbf{r}_E + \frac{1}{s} \mathbf{u} + \frac{1}{s} \mathbf{u} d\mu + \frac{1}{s} \mathbf{U} d\mathbf{a} \tag{3d}$$

In addition to the coordinate corrections $d\mathbf{x}$ for S, the system of observational equations also contains the four parameters ($d\mu$, $d\mathbf{a}$) of the linear transformation, i.e. altogether seven unknowns.

4.2. Determination with the aid of control points on the sea floor

If MG control points are available on the sea floor the space position of points on the sea surface or in the sea can be determined by acoustic (US) techniques. They make use of distances measured with US, directions or Doppler frequencies, and are distinguished by the kind of observation data and the type of measuring instruments. Designations and terms are often used that are not familiar to geodesists. For this reason, some terms used in MG have been compiled (see /6/).

Baseline — The line or separation between hydrophones, transponders or beacons comprising an array of acoustic devices used to determine the position of vehicle.
Continuous Wave (CW) *Beacon — An underwater device which continuously* emits an acoustic signal with precisely known frequency and phase characteristics (usually a single frequency sinusoid).
Pinger — An underwater device that repeatedly transmits acoustic pulses at a non-precise rate (approximately periodic).
Pulsed Beacon — An underwater device which repeatedly (usually periodically) emits an acoustic pulsed signal at precisely known times.
Responder — An underwater device that emits an acoustic pulsed signal upon receipt of an electrical trigger pulse via a wire connection.
Transponder — An underwater acoustic device which after receiving a proper acoustic pulsed interrogation signal, transmits generally a different acoustic pulsed signal delayed a precise and fixed period of time after receipt of the interrogation pulse.
Reply *diversity — In order to uniquely distinguish between many transponders,* the reply signals are usually different (for example different reply frequencies) while all the transponders in an array are interrogated by the same signal.

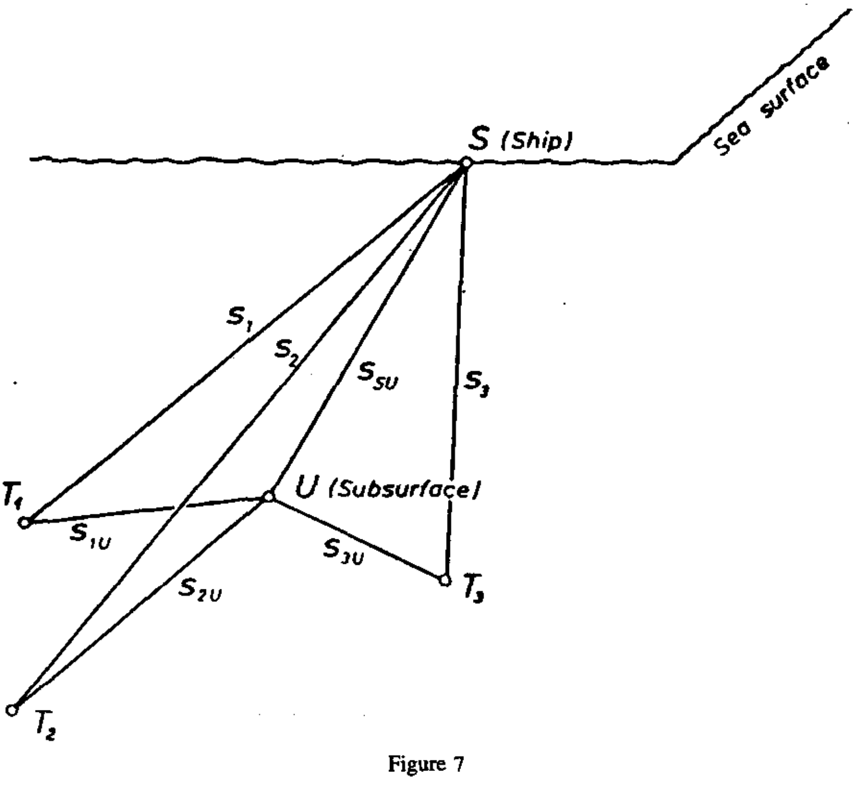


Figure 7

The techniques subsequently described aim at determining the space position of points on the sea surface from US measurements. They are differing in accuracy and correspond only in part to the requirements usually made in CG for determining control points. Their designation in MG is according to the observation data used and departs in most cases from the designation used in CG.

The *Range-Range Long Baseline* technique corresponds to the two- or three-dimensional resection by distances *(Ann. 1)*. US distances are measured to $i \geqslant 2$ MG control points C_i on the sea floor that are equipped with transponders or beacons. With known sea topography the position of the point S is defined from $i=2$ distances; for $i=3$ the space position can be determined without a prerequisite. For $i > 3$ there is redundancy in determination giving rise to adjustment. The measurement is made by determining US travel times and density profiles. With the use of transponders the number of simultaneously determinable ship positions is limited; with pulsed beacons this barrier is avoided. *Fig. 7* shows an array for the simultaneous determination of a surface point S and of a U situated in the sea (submerged vehicle) with the aid of three transponders. The resolving power of the measuring instruments is approximately $\pm$ 1 m, the position accuracy obtainable in the deep ocean is stated with $\pm$(5 to 20) m.

The *Range-Range Short Baseline* technique aims at determining the position and motion of a submerged vehicle U relative to a surface ship S. Three hydrophones H_i are used that are located in a horizontal plane below the survey ship (see *Fig. 8*). The vehicle to be determined is equipped with a beacon, transponder or responder. The relative position of U to the coordinate system of the three points H_i can be computed from the measured distances $\overline{UH_i} = s_{iu}$. To determine the motion of the system H_i connected with the ship the distances s_i to a fixed transponder on the sea floor are measured at the same time.

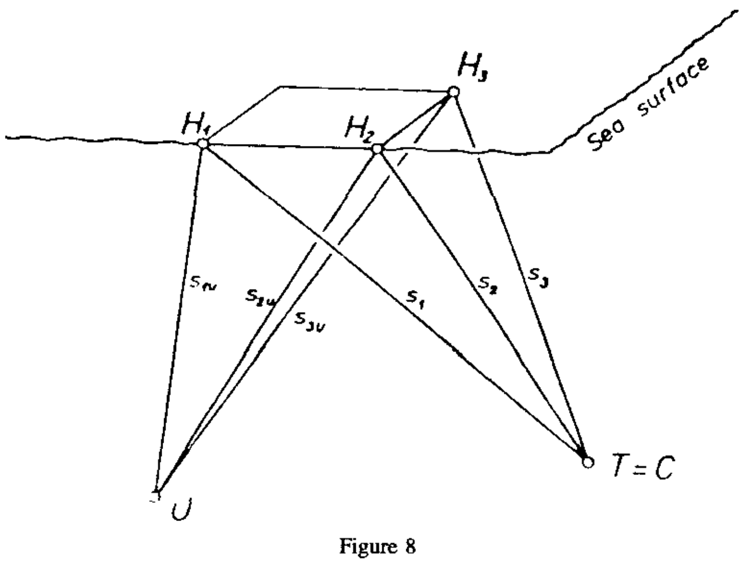


Figure 8

The *Time-Difference Short Baseline* technique is used to determine the motion of a ship relative to a fixed point C on the sea floor. A pinger installed in C sends out acoustic pulses. The travel times t_i and their differebces Δt are measured on three hydrophones H_i which are arranged in a plane connected with the ship (see *Fig. 9*). If $\overline{HC} = z$ denotes the depth difference, v the velocity of US, and b the distance of the hydrophones, and if z is much larger than b, the following approximate relation will hold:

$$\sin\beta = \frac{1}{b} v\Delta t$$
$$x = z\tan\beta \approx \frac{1}{b} zv\Delta t \tag{4}$$

In analogy, the second position coordinate y can be determined with the aid of the third hydrophone.
The *Range-Bearing* technique makes use of the polar determination familiar to geodesists by measuring direction and distance. In the control point C on the sea floor, a transponder is installed; hydrophones H_i are connected with the ship; their distance b_i is about one-third of the length that corresponds to the US frequency. Two hydrophones H, H_1 are horizontal, H, H_2, however, vertical (see *Fig. 10*). The travel time t of the US wave emitted from C and the phase differences φ_1, φ_2 as against the US waves arriving at H_1, H_2 are measured in H. If the distance b

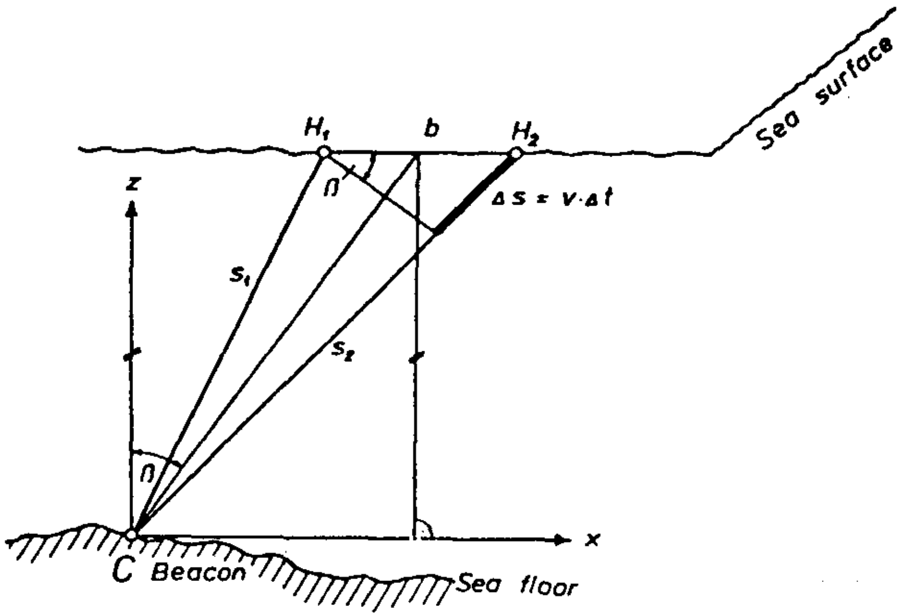


Figure 9

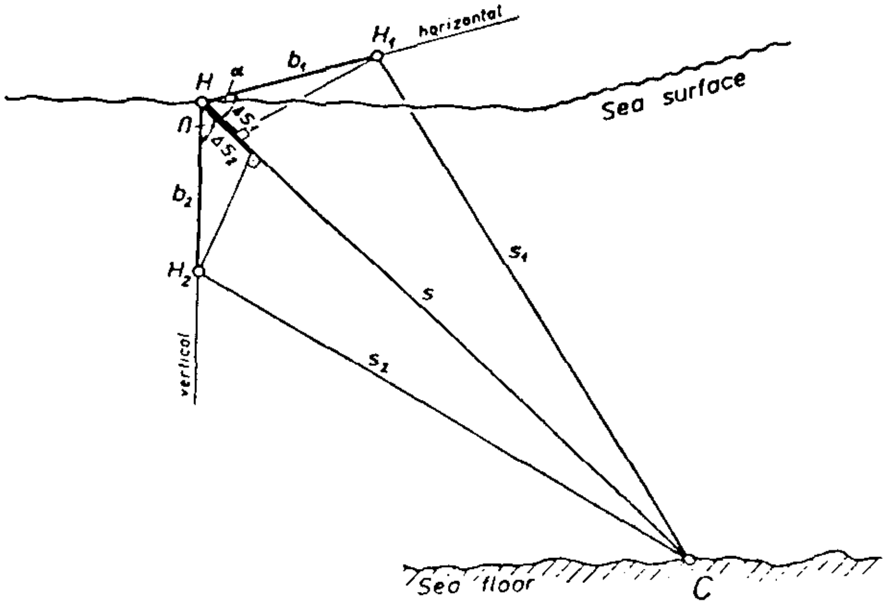


Figure 10

of the hydrophones is sufficiently small as against the distance s the following relations hold:

$$\Delta s_1 = \frac{\varphi_1}{2\pi}\lambda = b_1\cos\alpha \ , \quad s_2 = \frac{\varphi_2}{2\pi}\lambda = b_2\cos\beta \qquad (5)$$

The coordinates (α,β) of the direction (HC) in the system given by H_i can be computed therefrom. The position of H to C is defined by (α,β,s). The accuracy of the direction determination is poor (about $\pm 3°$); an accuracy of $\pm$(5 to 20) m is indicated for the determination of the space position of H.

Also the position determination by two acoustically measured oriented directions is being used (*Bearing-Bearing* technique). The precision is poor and the application limited.

An acoustic version of the electromagnetic LORAN method is the *Range-Range Hyperbolic* technique. Simultaneous signals are emitted from two beacons C_1, C_2 on the sea floor, and the difference Δt of their travel times is measured in the ship position S to be determined. Δt defines a hyperboloid with C_1, C_2 as focal points. With known surface topography the position of S is defined by two measurements Δt_i; at least three beacons are required. A determination without any prerequisite demands the measurement of at least three values Δt_i, i.e. the installation of three pairs of independent beacons.

Also *acoustic Doppler* systems are being used. As with electromagnetic waves, distinction is made between continuous and pulse techniques. The former use three

bottom-moored CW beacons C_i emitting US waves with various frequencies. By integration of the received Doppler frequencies f_D in the moving ship position S, the variations ds_i of the distances $\bar{S}\hat{C} = s$ are measured which take place due to the motion of S *(Fig. 11a)*.
The following relations hold:

$$ds_i = \lambda f_{Di} = -\frac{v}{f} f_{Di} \tag{6}$$

λ length, v velocity, f frequency — of the US wave

From the equations holding according to Annex 2 between the distance differences ds_i, the variation $\mathbf{dx}$ of the coordinate vector of S, and the direction $\mathbf{r}_i = (SC_i)$

$$ds_i = \mathbf{r}_i^T \, \mathbf{dx} \tag{7a}$$

$\mathbf{dx}$ can be computed (see Annex 3)

$$\mathbf{dx} = \sum_{i=1}^{3} \mathbf{r}_i^* \, ds_i \tag{7b}$$

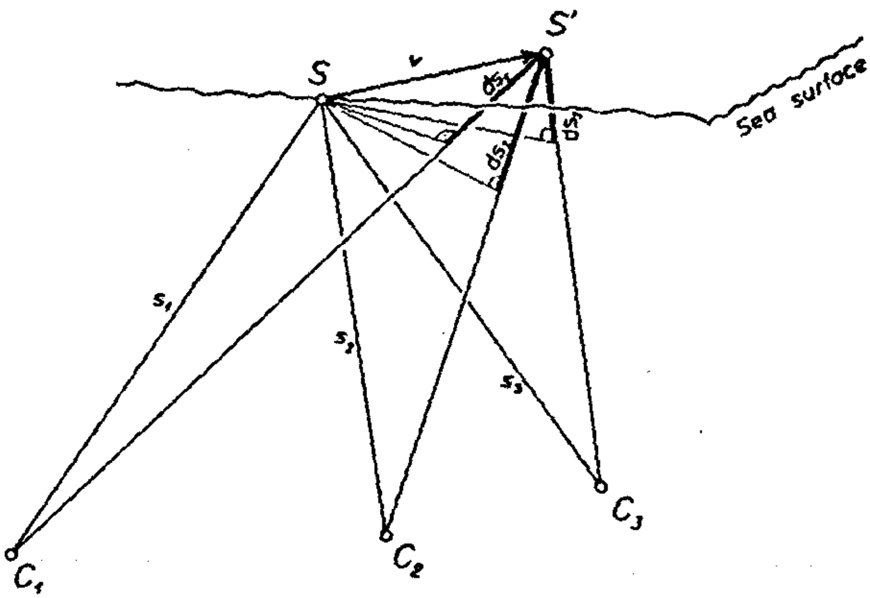


Figure 11a

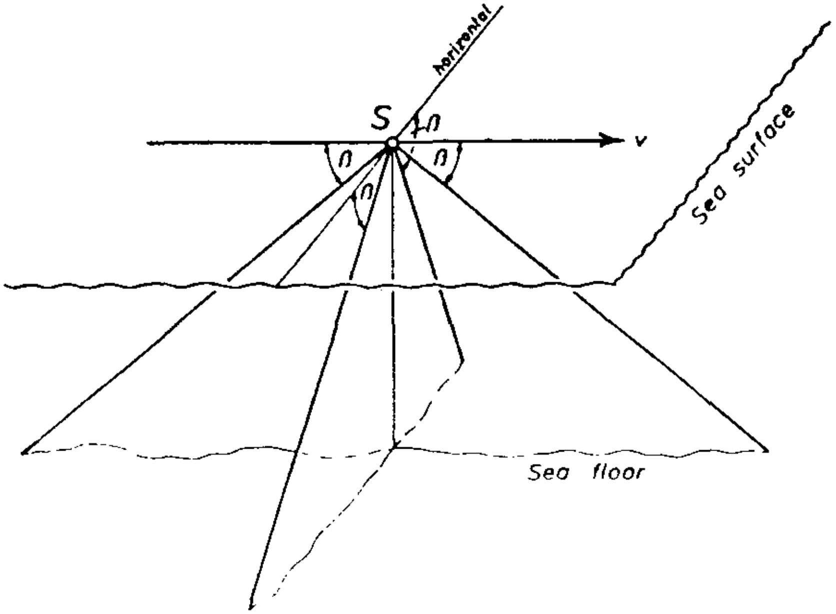


Figure 11b

The precision in determining the coordinate variations is indicated with ± 10 cm. The Doppler pulse technique corresponds to the Doppler navigation known from aviation. From a transducer located at a ship position S, US pulses are emitted under a predetermined angle β (see *Fig. 11b*). After reflection from the sea floor the frequency variation due to the ship's motion is integrated. If f_D denotes the integral per unit time and v_s the velocity of motion of S, the following relation holds:

$$f_D = -\frac{2v}{v_s} f \cos \beta \tag{8}$$

If four simultaneous measurements are made in the direction opposite to the ship's motion and normal thereto, the ship's velocity will follow from the mean value.

5. Topographic survey of the sea floor

The making of topographic charts of the sea bottom (bathymetric maps) is an important task of MG the importance of which increases with the economic exploitation of the oceans. Topographic sea maps describe the sea floor by means of contours and depth figures. They contain the MG control points, all data important for navigation (directions, baselines, zones) and other remarkable natural and artificial realities such as mineral deposits, properties of the sea bottom, etc. The maps are made in a unified grid, a unified geodetic datum and a unified mathematic projection. The latter is chosen so that navigation data may be entered or taken without any essential reduction. The map scale will be chosen according to the economic importance of the respective area.

Of particular importance at present are the continental shelf regions, i.e. the parts of continents covered by the sea to a depth of about 200 m (see /7/). From the shelf areas the continents are rising, the ocean bottom falls steeply down to the deep sea. Petroleum and natural gas are presumed to exist in the shallow shelf regions that are mostly covered by sediments; exploration is in full swing. According to an estimate (see /6/) petroleum yields of 130 billion tons may be expected in promising areas of abt. 6.0 million square kilometres. This is added by areas of abt. 13 million square kilometres with moderate chances of yield. With a present world production of abt. 2.5 billion tons per year, oil reserves for abt. 50 to 100 years may be expected from the shelf regions.

For the shelf areas, topographic maps are demanded at scales of 1:25,000 to 1:250,000. Since their surface is about 15% of the surface of all continents this will mean an enormous geodetic task. For the open sea covering abt. two-thirds of the entire earth's surface, scales as from 1:1,000,000 are considered to be convenient.

The standard method for surveying the sea floor is the determination of polar coordinates with reference to a survey ship. The position of the survey ship is defined by navigation techniques controlled and corrected with the aid of MG control points on the sea floor, on land or in satellites with known ephemerides (integrated systems).

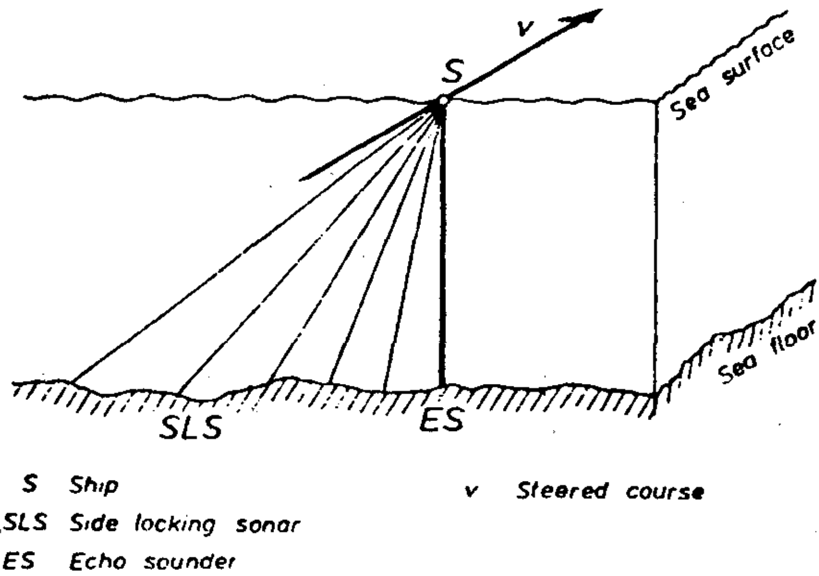


Figure 12a

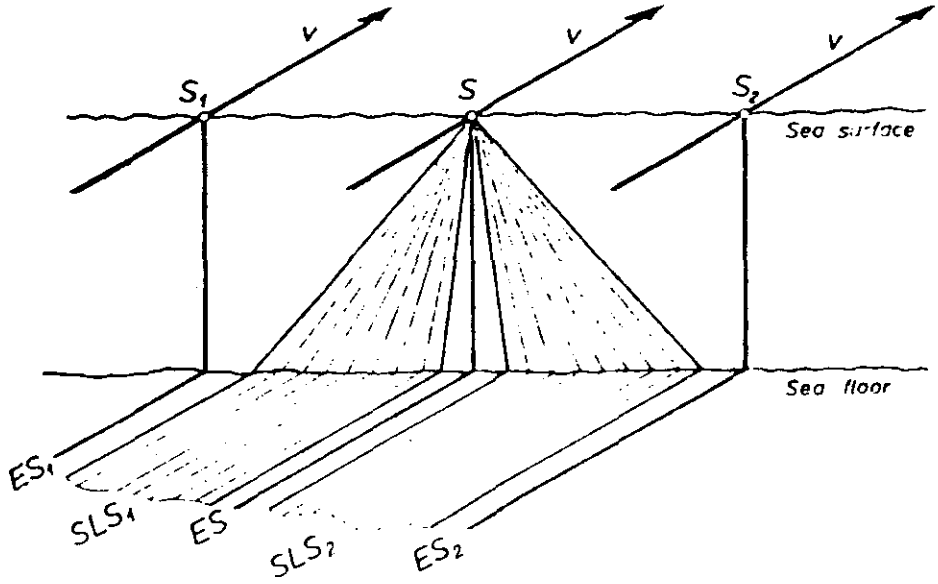


Figure 12b

The simplest case is echo sounding with vertical surveying direction. To increase the economy of this «locus navigation», side-looking sonar profiles are taken at the same time. These are added by parallel profiles from auxiliary vessels whose position with respect to the main vessel is defined by electromagnetic distance measurement. Main and auxiliary vessels form a multibeam echo sounding system covering with high precision a wide strip of the sea floor *(Fig. 12)*.

The course of this measuring system is defined by integrated navigation techniques (see /8/). That means that in addition to the navigation technique proper (inertial systems, Loran, etc.) data to control points (in the sea, on the shore or in space) are observed at certain times t. Each measured value defines one (distances) or two (directions) observational equations at the time of measurement t. For the position vectors $\mathbf{x}(t_i)$, $\mathbf{x}(t_j)$ of the measuring points belonging to t_i and t_j a condition follows from the equation $\mathbf{x} = \mathbf{F}(t,p_k)$ for the ship's course. A linear transformation may be introduced to correct the distances (velocity) and the orientation (course) (cf. Eq. (3), Sect. 4).

$$\Delta\mathbf{x}(t_i,t_j) = \mu\mathbf{R}\Delta\mathbf{F}(t_i,t_j,p_k)$$

μ scale factor
$\mathbf{R}$ orientation matrix
p_k course parameter

In exceptional cases the sea floor may be surveyed by photogrammetric methods. With small water depths to abt. 10 m it is possible to determine the sea *bottom by means of aerial photographs. For this purpose the techniques of two-*media photogrammetry have been developed in which the refraction of light beams on the sea surface is taken into account. In greater depths the survey can be made from submarines while the object to be surveyed is illuminated. This may be of importance in special cases such as the search for wreckage. Detailed surveys of the sea floor are also carried out from submarines by means of US profiles. The position of the submarine is defined by one of the described positioning techniques from the surface ship during the observation, or may be derived from MG control points on the sea bottom.

6. DETERMINATION OF THE SEA SURFACE AND THE GEOID

For geodesy the shape of the earth's potential surface is important that co*rresponds to the mean sea level and is termed «geoid». Oceanography is interested* in the topography of the mean sea surface itself (sea topography). This is the fictitious surface the sea surface would accept if all shortand long-termed motions due to tides and winds were eliminated.

From theoretical reflections it is known that these two surfaces do not coincide and vary in height by amounts of up to 2 m approximately. This leads to the noteworthy effect that geometric levelings along shore lines give results different from the observation of long-year mean water level readings. Since, however, the national geodetic altitude systems are derived from water level readings this results in transformation problems in the connection or combination of these systems.

Oceanography has set up theories according to which the height differences between the two surfaces can be computed when the density, the salt content and the sea currents are known; nevertheless, the practical results do not fully agree with these theories. If the differences are known it is possible to determine the geoid from sea topography, and vice versa.

At the present state of measuring techniques the difference between the two surfaces is insignificant for global reflections and can therefore be neglected. Thus, *sea topography is at present a good approximation for the geoid, and vice versa.* Determination of the sea topography or of the geoid is a task of high importance and topicality both for geodesy and oceanography.

6.1. Sea topography

Sea topography cannot be determined with the methods of classical geodesy. Only with the aid of satellite altimetry this task can be carried out with the required generality and accuracy. In these techniques, a radar altimeter is installed in the satellite that emits microwaves in a narrow cone of 0.1° aperture and vertical axix. They are reflected from the sea surface and sent back to the satellite. From the travel time difference between emitted and reflected wave results the distance of the satellite from the sea surface. The distance ΔH measured in the vertical from the satellite to the undisturbed sea surface (sea topography) is obtained by filtering out the periodic disturbances and averaging all reflections occurring in the region of the section of the radiation cone with the sea (footprints) (see *Fig. 13*). The accuracy of determination at present lies at about ±(1 to 2) m, for the next few years an increase to ±0.1 m may be expected (see /9/).

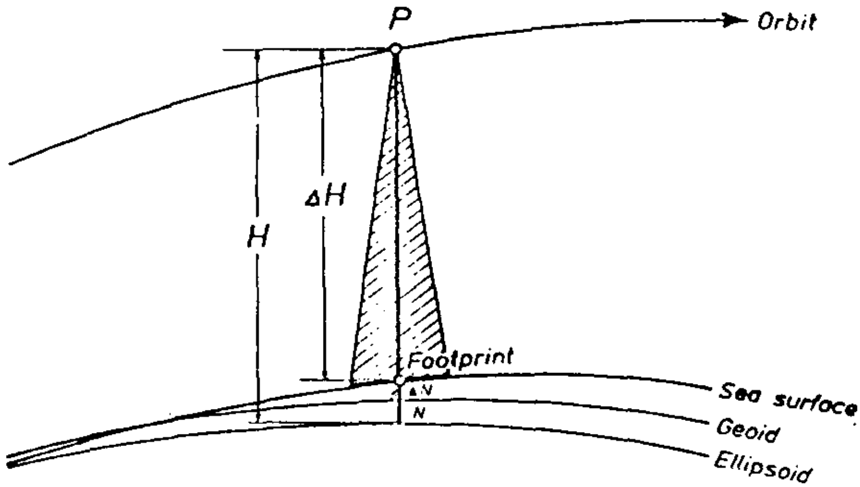


Figure 13

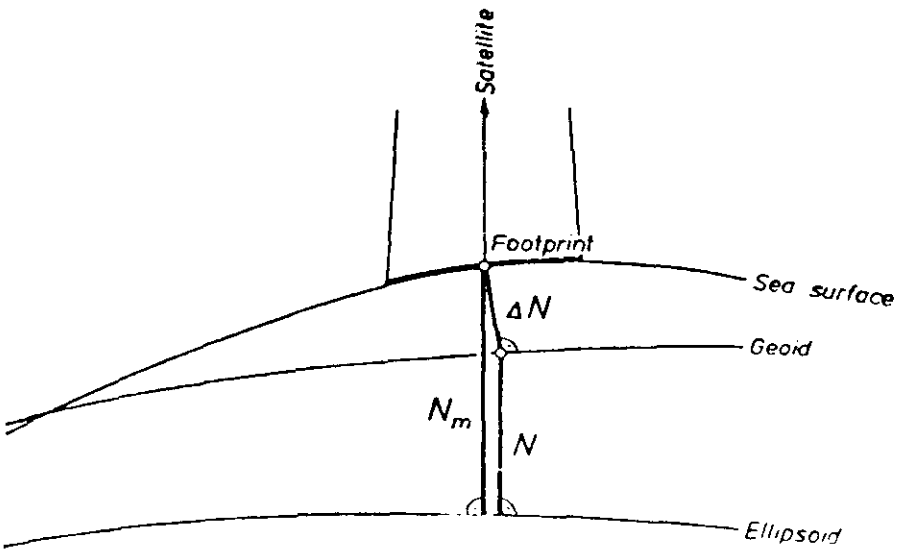


Figure 13 (Detail)

The distance H of the satellite from the reference ellipsoid (height H) can be computed from its known ephemerides determined separately. The difference as against the altimeter value $\triangle H$ gives the ellipsoidal height (undulation) N_m of the undisturbed sea surface.

Satellite altimetry was successfully tested in the Skylab experiment (see /10/). GEOS C is equipped with an altimeter providing distances with a precision of abt. (1 to 2) m at a footprint diameter of abt. 8 km. Since in the course of a year every square degree of the oceans is passed about 10 times by GEOS C and measurements are made at one-second intervals the resulting mean values will be free from short-and long-termed influences and correspond to the topography of the quasistationary undisturbed sea. As from 1978 the launching of the SEA-SAT series is planned that will carry improved altimeters with an accuracy of abt. ± 0.1 m. On the basis of the resulting measuring data it will be possible to distinguish the sea topography from the geoid.

6.2. **Geoid**

On principle, the dynamic and astrogeodetic techniques of terrestrial and satellite geodesy used on land can be applied to determining the geoid above the sea. Determination of the required observation data on the sea is more difficult and inaccurate because of the motion of the sea surface. On the other hand, reduction problems are largely avoided as the sea surface nearly coincides with the geoid.

From the dynamic satellite technique follow the rough structures of the field of gravity and of the geoid above the sea. To determine the fine structures, gravity measurements or astronomic measurements must be carried out on the sea itself. Great difficulties arise due to the motion of the observing stations.

With the sea gravimeter it is possible to determine gravity values on ships with a maximum precision of ± 1 m Gal. This is 100 times worse than on land (see /11/). In gravity measurements on the sea floor or in submarines the motion problems are partly omitted. Additional difficulties will, however, arise from the necessary remote recording and the placement of equipment. The distances N of the geoid from the reference ellipsoid (undulations) can be derived from the determined gravity anomalies $\triangle g$ according to Stokes' formula.

$$N_o = \frac{1}{4R\pi\gamma} \iint_\sigma \triangle g S(\psi)\, d\sigma \qquad (1)$$

R, γ mean value: gravity and earth's radius
$S(\psi)$ Stokes' function
σ surface

The accuracy of these values is about two tenth powers lower than on land. For the astronomic-geodetic determination of the geoid, deflections from the vertical would have to be determined by astronomic measurement. This can be done with suitable photographic cameras (see /12/). The accuracy obtainable is again poor; it is abt. ±16″ as against ±0.5″ on land. Besides, the procedure is expensive and time-consuming. Therefore it can only be applied in special cases and must be disregarded for general use. The determination of fine structures of the field of gravity and of the geoid cannot be carried out by the classical methods used on land. At present this is only possible by determining the sea topography with the aid of satellite altimetry. With known shape of the geoid, i.e. of the N values, the gravity anomaly Δg can be computed by inverting Stokes' formula.

$$\Delta g_o = -\frac{N_o \gamma}{R} - \frac{\gamma}{16\pi R} \iint_\sigma \frac{N - N_o}{\sin^3\left(\frac{\psi}{2}\right)} d\sigma$$

Knowledge of the gravity anomalies and of the deflections from the vertical is an important prerequisite for the improvement of navigation techniques, for the geophysical exploration and for scientific studies on the state and motion of the earth's crust below the sea.

7. OUTLOOK TO FUTURE DEVELOPMENTS

With the increasing world population and increasing living standards the demand for foodstuffs, energy and raw materials will also increase steadily. To cover this demand the reserves situated in the sea and in the earth's crust below the sea will have to be exploited to an increasing extent. This will mean an extension of terrestrial methods for technical, economic and ecological development to the oceans. Since two-thirds of the entire earth's surface are involved the task to be carried out will be enormous.

For geodesy this means the necessity of developing the techniques of marine geodesy and of providing in due time, just as on land, the bases for projects, economic decisions and technical operations. Besides, MG will have to ascertain the longtermed geodynamic variations of the earth in the region of the world oceans and thus provide the prerequisites for initiating measures against approaching catastrophes, i.e. for protecting our environment.

According to the statements made in the preceding chapters MG will have to be concerned with the following tasks, also in the future:
Position determination on the sea, in the sea and on the sea floor, production of topographic and thematic charts of the sea bottom, determination of the sea surface topography and of the sea geoid, determination of the geodynamic variations of the earth's crust, of the geoid and of the sea topography, establishment of test areas for testing instruments and techniques, conclusion of international agreements for carrying out experiments and practical tasks. Solutions already existing and solution setups are to be further developed to increase their accuracy, complete

their statement and reduce the time required therefor. Hence follow actual guiding principles to carry on the tasks formulated.

For position determination: Creation of global nets of MG control points on the sea floor (with four transponders each in optimum arrangement), development of instruments and techniques for an accurate satellite (Range-Rate) navigation and for US trilaterations in the sea. Systematic investigation of the techniques of terrestrial and satellite navigation for the position determination on and in the sea. Combination of individual techniques to integrated techniques of highest precision taking account of the required datum transformations. Application of the results to the creation of a numerical marine cadastre for the delimitation of sea lots and the safeguarding of legal rights (servitudes).

In the production of maps: Production of topographic maps at scales as from 1:25,000 for the shelf regions and the deep sea by echo sounding and side-looking sonar profiles. Investigation of techniques for profile taking and position determination from submerged vehicles, development of survey techniques in the sea with certain optical and micro waves and US holography. Investigation of propagation conditions of EM and US waves in the water. Application of two- and multi-media photogrammetry to the survey of objects in special cases. Development of remote sensing techniques in the sea.

For determination of the sea topography and of the geoid: Determination of the sea surface topography by satellite altimetry with a precision of ± 0.1 m referred to the datum of the satellite orbit. Determination of the marine geoid from astronomic and gravity measurements on and in the sea by the methods of physical and astronomic geodesy in conjunction with techniques of dynamic satellite geodesy. Condensation of gravity profiles on and in the sea.
Study of the reduction problems on hand. Determination of the distance between the undisturbed sea surface and the geoid from oceanographic theories. Comparison of geoid surfaces following from altimetry and terrestrial techniques. Determination of the microstructures of the field of gravity from the undulations of the geoid by inverting Stokes' formula.

For geophysical and geodynamic research: Development of techniques for finding minerals on the sea floor and in the crust thereunder. Derivation of statements, on the basis of measurements, on time-dependent variations of the shape of the earth's crust, of the parameters of the geoid and of the fields of gravity. Determination of minimum motions for the early detection of seaquakes (Tsunami) and other catastrophes, establishment of an automatic warning system therefor.

The aims for the further development of MG have also been expressed in the recommendations given at the MG symposia of recent years. In Columbus in 1974 the following was recommended:

(1) (a) Establishment of marine test sites or marine geodetic ranges suitable for the calibration and verification of precise geodetic and navigation methods,

(b) Conduct of «controlled-condition» experiments to determine the best accuracy attainable,
(c) Publication of results of such experiments.

(2) Continued studies on methods of determining the form of the geoid by satellite altimetry, taking oceanographic parameters into account.
(3) Design and conduct of specific marine geodetic experiments and development of improved data-analysis techniques, to permit determination of tides, mean sea level, ocean surface heights above mean sea level, and sea floor spreading.
(4) Development of astronomic instruments usable at sea to make possible geoid determination by astrogravimetric and astro-satellite techniques.
(5) Establishment of geometric geodetic systems for determining marine boundaries and for positioning based upon existing technology, with the capability for improvement in accuracy with increasing need and improved technology.
(6) Use of SI units in all reports, at least parenthetically.
(7) Establishment of communication with users relative to their needs and accuracy requirements.
(8) Development of space systems that would make possible better position determination at sea.
(9) Continuation of efforts to increase the accuracy of position determination at sea by the optimum integration of different navigational aids.

Summarizing the following can be said: MG is a young discipline, its roots lying in oceanography and geodesy are, however, very old. The practical importance of MG is due to the necessity of integrating the oceans in our living space and of developing technical and geodetic operations on, in and on the bottom of the sea. Moreover MG has become an essential component of the geodetic discipline that will contribute to answer the old geodetic question for the geometric shape of earth, the structure of its field of gravity and its variations with respect to time, with higher accuracy, completeness and within shorter time. Therefore MG will have to be included in the future training of geodesists at universities and recognized as a separate section of geodesy.

ANNEX 1. SPACE RESECTION WITH DISTANCES

1. PROBLEM

given : the coordinate vectors $\mathbf{x}_i$ of 3 points C_i
measured : the 3 distances $s_i = \overline{C_iS}$ from a point S to C_i
wanted : the coordinate vector $\mathbf{x}_s$ of S

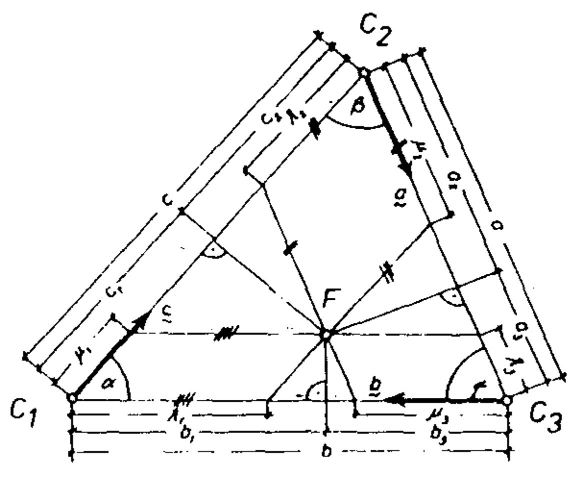
2. GENERAL SOLUTION

S is the point in which 3 spheres with center C_i and radius s_i intersect. There are *two solutions* S_1 *and* S_2 *which are symmetrical to the plane* $(C_1C_2C_3)$.
The equations for $\mathbf{x}_s$

$$(\mathbf{x}_S - \mathbf{x}_i)^2 = s_i^2 \qquad i = 1, 2, 3 \tag{1}$$

are not linear. Therefore a transformation has to be made to get the general solution. Following [Rinner, JEK VI, pp 699] this can be done in the following several times controlled steps.

2.1. Calculation of the sides, directions and angles of the triangle C_1, C_2, C_3



sides:

$$a = \overline{C^2C^3} = |\mathbf{x}^3 - \mathbf{x}^2| \quad b = \overline{C^3C^1} = |\mathbf{x}^1 - \mathbf{x}^3| \quad c = \overline{C^1C^2} = |\mathbf{x}^2 - \mathbf{x}^1| \tag{2a}$$

directions:

$$\mathbf{a} = \frac{1}{a}(\mathbf{x}_3 - \mathbf{x}_2) \quad \mathbf{b} = \frac{1}{b}(\mathbf{x}_1 - \mathbf{x}_3) \quad \mathbf{c} = \frac{1}{c}(\mathbf{x}_2 - \mathbf{x}_1) \tag{2b}$$

$$\text{proof: } \mathbf{a}^2 = \mathbf{b}^2 = \mathbf{c}^2 = 1$$

angles:

$$\begin{aligned} \cos\alpha &= -\mathbf{b}^T\mathbf{c} & \cos\beta &= -\mathbf{c}^T\mathbf{a} & \cos\gamma &= -\mathbf{a}^T\mathbf{b} \\ \sin\alpha &= |\mathbf{Bc}| & \sin\beta &= |\mathbf{Ca}| & \sin\gamma &= |\mathbf{Ab}| \end{aligned} \qquad (2c)$$

proof: $\alpha + \beta + \gamma = 180^\circ$

remark

$|\mathbf{x}| = \sqrt{x^2 + y^2 + z^2}$ absolute value (scalar) of $\mathbf{x}$

$\mathbf{X} = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix}$ axiator of $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$\mathbf{Xy} = \mathbf{x} \times \mathbf{y}$ vector product

2.2. Calculation of the coordinate vector $\mathbf{x}_F$ of the intersection point F of the line (S_1, S_2) with the plane (C_1, C_2, C_3)

a. *auxiliar parameters*

$$\begin{aligned} a_2 &= \frac{1}{2\,a}(a^2 + s_2^2 - s_3^2) & a_3 &= \frac{1}{2\,a}(a^2 - s_2^2 + s_3^2) \\ b_3 &= \frac{1}{2\,b}(b^2 + s_3^2 - s_1^2) & b_1 &= \frac{1}{2\,b}(b^2 - s_3^2 + s_1^2) \\ c_1 &= \frac{1}{2\,c}(c^2 + s_1^2 - s_2^2) & c_2 &= \frac{1}{2\,c}(c^2 - s_1^2 + s_2^2) \end{aligned} \qquad (3)$$

$$a_2 + a_3 = a, \quad b_1 + b_3 = b, \quad c_1 + c_2 = c$$

b. *coordinates of F in the system $C_1\ C_2\ C_3$*

$$\begin{aligned} \lambda_1 &= \frac{1}{\sin^2\alpha}(b_1 - c_1\cos\alpha) & \mu_1 &= \frac{1}{\sin^2\alpha}(c_1 - b_1\cos\alpha) \\ \lambda_2 &= \frac{1}{\sin^2\beta}(c_2 - a_2\cos\beta) & \mu_2 &= \frac{1}{\sin^2\beta}(a_2 - c_2\cos\beta) \\ \lambda_3 &= \frac{1}{\sin^2\gamma}(a_3\ b_3\cos\lambda) & \mu_3 &= \frac{1}{\sin^2\lambda}(b_3 - a_3\cos\gamma) \end{aligned} \qquad (4)$$

proff: $\lambda_1 : \mu_2 = b : a, \quad \lambda_2 : \mu_3 = c : b, \quad \lambda_3 : \mu_1 = a : c$

$$\lambda_1\,\lambda_2\,\lambda_3 = \mu_1\,\mu_2\,\mu_3$$

c. *coordinate vector* $\mathbf{x}_F$

$$\begin{aligned} \mathbf{x}_F &= \mathbf{x}_1 - \lambda_1\mathbf{b} + \mu_1\mathbf{c} = \\ &= \mathbf{x}_2 - \lambda_2\mathbf{c} + \mu_2\mathbf{a} = \\ &= \mathbf{x}^3 - \lambda^3\mathbf{a} + \mu^3\mathbf{b} \end{aligned} \tag{5}$$

2.3. Calculation of $\mathbf{x}_{S_1}$, $\mathbf{x}_{S_2}$

a. *vector* $\boldsymbol{n}$ *= (FS)*

direction: $$\mathbf{r}_n = \frac{1}{\sin\gamma}\mathbf{Ab} = \frac{1}{\sin\alpha}\mathbf{Bc} = \frac{1}{\sin\beta}\mathbf{Ca} \tag{6a}$$

distance: $$s_n = \sqrt{s_1^2 - (\mathbf{x}_F - \mathbf{x}_1)^2} = \sqrt{s_2^2 - (\mathbf{x}_F - \mathbf{x}_2)^2} = \sqrt{s_3^2 - (\mathbf{x}_F - \mathbf{x}_3)^2} \tag{6b}$$

b. *coordinate vector*

$$\mathbf{x}^{S_{1,2}} = \mathbf{x}_F \pm s^n\mathbf{r}^n \tag{7}$$

3. LEAST SQUARE ADJUSTMENT

If $i > 3$ distances s_i to known points C_i are measured an approximate vector $\mathbf{x}_s'$ is calculated from 3 distances with the formulas of section 2. This vector will then be improved using least square technique of variation of coordinates.
Be s_i' the distances calculated with $\mathbf{x}_s'$ using equation (1), v_i the improvement of the measured distances s_i and

$$ds_i = \frac{1}{s_i}(\mathbf{x}_s' - \mathbf{x}_i)^T d\mathbf{x}_s = \mathbf{r}_i^T d\mathbf{x}_s.$$

Each distance yields an equation

$$s_i + V_i = s_i' + ds_i$$

and from these the system of observation equations follows:

$$\mathbf{v} = \mathbf{A}d\mathbf{x}_s - \mathbf{l} \tag{8a}$$

$$\mathbf{v} = (v_1 \ldots v_n)^T, \quad \mathbf{A} = [\mathbf{r}_1 \ldots \mathbf{r}_n]^T, \quad \mathbf{l} = (s_1'-s_1 \ldots s_n'-s_n)^T$$

Introducing the weight matrice $\mathbf{P}$ for the distances s_i we get the normal equation

$$\mathbf{A}^T\mathbf{P}\mathbf{A}d\mathbf{x}_s - \mathbf{A}^T\mathbf{P}\mathbf{l} = \mathbf{O} \tag{8b}$$

with solution

$$d\mathbf{x}_s = \mathbf{Q}\mathbf{A}^T\mathbf{P}\mathbf{l},\ \mathbf{Q} = (\mathbf{A}^T\mathbf{P}\mathbf{A})^{-1}$$
$$\mathbf{x}_s = \mathbf{x}_s' + d\mathbf{x}_s$$

ANNEX 2. DIFFERENTIAL FORMULAS

1. DISTANCES

$$s_{ij} = \overline{P_i P_j}$$

$$s_{ij}^2 = (\mathbf{x}_j - \mathbf{x}_i)^2, \qquad \mathbf{x}^T = (x,y,z) \tag{1a}$$

$$ds_{ij} = \mathbf{r}_{ij}^T(d\mathbf{x}_j - d\mathbf{x}_i) \tag{1b}$$

$$\mathbf{r}_{ij} = \frac{1}{s_{ij}} (\mathbf{x}_j - \mathbf{x}_i) = (u,v,w)_{ij}^T \tag{1c}$$

$$\mathbf{r}_{ij}^2 = (u^2 + v^2 + w^2)_{ij} = 1$$

2. DIRECTIONS

$$\mathbf{r}_{ij} = (P_i\, P_j)$$

2.1. Formulas with direction cosines

$$\mathbf{r}_{ij} = \frac{1}{s_{ij}} (\mathbf{x}_j - \mathbf{x}_i) = (u,v,w)_{ij}^T \tag{2a}$$

$$d\mathbf{r}_{ij} = \frac{1}{s_{ij}} (d\mathbf{x}_j - d\mathbf{x}_i) - \frac{1}{s_{ij}^2} (\mathbf{x}_j - \mathbf{x}_i) ds_{ij}$$

Introducing ds_{ij} from equ. (1b) we get:

$$d\mathbf{r}_{ij} = \frac{1}{s_{ij}} (\mathbf{E} - \mathbf{r}_{ij}\mathbf{r}_{ij}^T)\, (d\mathbf{x}_j - d\mathbf{x}_i) \tag{2b}$$

$$(\mathbf{E} - \mathbf{r}_{ij}\mathbf{r}_{ij}^T) = \mathbf{M}_{ij} = \begin{bmatrix} 1 - u^2 & -uv & -uw \\ -uv & 1 - v^2 & -vw \\ -uw & -vw & 1 - w^2 \end{bmatrix}_{ij}$$

$$d\mathbf{r}_{ij} = (du,\, dv,\, dw)_{ij}^T$$

2.2. Formulas for astronomic coordinates

δ declination
t hour angle

$$\mathbf{r}_{ij} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} \cos\delta \cos t \\ \cos\delta \sin t \\ \sin\delta \end{bmatrix} \tag{3a}$$

$$d\mathbf{r}_{ij} = \mathbf{A}_{ij} d\sigma_{ij}$$

$$\mathbf{A} = \begin{bmatrix} -\sin\delta \cos t & -\sin t \\ -\sin\delta \sin t & \cos t \\ \cos\delta & 0 \end{bmatrix}, \quad d\sigma = \begin{bmatrix} d\delta \\ \cos\delta\ dt \end{bmatrix} \tag{3b}$$

Because of the relation $\mathbf{A}^T\mathbf{A} = \mathbf{E}$ the inversion yields

$$d\sigma_{ij} = \mathbf{A}_{ij}^T\, d\mathbf{r}_{ij} = \frac{1}{s_{ij}} \mathbf{A}_{ij}^T\, \mathbf{M}_{ij}\, (d\mathbf{x}_j - d\mathbf{x}_i) \tag{3c}$$

2.3. *Calculation of the covariance matrix* $\mathbf{Q}_\sigma$

From the orientation procedure of the photographic plate we get the parameters

$\mathbf{o}^T = (x_o', y_o', c)$	inner orientation
$\mathbf{R} = \mathbf{R}(\alpha_1, \alpha_2, \alpha_3)$	orientation matrix

and their covariance matrix

$$\mathbf{z} = \mathbf{z}(x_o', y_o', c, \alpha_1, \alpha_2, \alpha_3),\ \mathbf{Q}_z \tag{4a}$$

Be $\mathbf{x}'$ image (plate) coordinates and $\mathbf{x}_o'$ the principal point of the plate in the local plate system. The local direction

$$\mathbf{r}' = \frac{1}{s'} (\mathbf{x}' - \mathbf{x}_o') \tag{4b}$$

is transformed into the astronomic system by the matrix $\mathbf{R}$

$$\mathbf{r} = \mathbf{R}\mathbf{r}' \tag{4c}$$

Differentiating this equation and using equ. (2b) we get

$$d\mathbf{r} = d\mathbf{R}\mathbf{r}' + \frac{1}{s'} (\mathbf{E} - \mathbf{r}'\mathbf{r}'^T)\,(d\mathbf{x}' - d\mathbf{x}_o') \tag{4d}$$

Since $d\mathbf{R}$ is a function of $(d\alpha_1, d\alpha_2, d\alpha_3)$ and $d\mathbf{x}_o$ a function of (dx_o, dy_o, dc) equ. (4d) is a linear function for $\mathbf{dz}$ (see equ. (4a))!

$$\mathbf{dr} = \mathbf{Hdz} \tag{4e}$$

Introducing above equation into equ. (3c)

$$d\sigma = \mathbf{A}^T\mathbf{Hdz}$$

the matrix $\mathbf{Q}_\sigma$ can be calculated using the law of error propagation:

$$\mathbf{Q}_\sigma = (\mathbf{A}^T\mathbf{H})\mathbf{Q}_z(\mathbf{A}^T\mathbf{H})^T$$

With $t' = t\cos\delta$ as auxiliar parameter it follows

$$\mathbf{Q}_\sigma = \begin{bmatrix} Q_{\delta\delta} & Q_{\delta t'} \\ Q_{t'\delta} & Q_{t't'} \end{bmatrix} \qquad \begin{aligned} Q_{t'\delta} &= \cos\delta\, Q_{t\delta} \\ Q_{t't'} &= \cos^2\delta\, Q_{tt} \end{aligned}$$

2.4. Properties of the matrices A, M, R

$$\mathbf{A}^T\mathbf{A} = \mathbf{E}$$

$$\mathbf{A}\mathbf{A}^T = \mathbf{E} - \mathbf{r}\mathbf{r}^T = \mathbf{M}$$

$$\mathbf{R}_s\mathbf{M} = \mathbf{r}$$

$$\mathbf{R}_s^2 = -\mathbf{M} \qquad \mathbf{R}_s = \begin{bmatrix} 0 & w & -v \\ -w & 0 & u \\ v & -u & 0 \end{bmatrix} \quad \text{Axiator}$$

ANNEX 3. RECIPROCAL VECTORS

1. DEFINITION

a, b, c arbitrary 3D-vectors (not complanar)
a*, b*, c* reciprocal vectors

$$\mathbf{a}^* = \frac{\mathbf{b} \times \mathbf{c}}{(\mathbf{a}, \mathbf{b}, \mathbf{c})} \qquad \mathbf{b}^* = \frac{\mathbf{c} \times \mathbf{a}}{(\mathbf{a}, \mathbf{b}, \mathbf{c})} \qquad \mathbf{c}^* = \frac{\mathbf{a} \times \mathbf{b}}{(\mathbf{a},\mathbf{b},\mathbf{c})} \qquad (1a)$$

$\mathbf{a} \times \mathbf{b}$ vector product

$(\mathbf{a},\mathbf{b},\mathbf{c}) =$ det($\mathbf{a},\mathbf{b},\mathbf{c}$)

$$\begin{aligned} \mathbf{a}^T\mathbf{a}^* = \mathbf{b}^T\mathbf{b}^* = \mathbf{c}^T\mathbf{c}^* = 1 \\ \mathbf{a}^T\mathbf{b}^* = \mathbf{a}^T\mathbf{c}^* = \mathbf{b}^T\mathbf{c}^* = 0 \end{aligned} \qquad (1b)$$

Any vector **x** can be expressed by **a**, **b**, **c** and **a***,**b***,**c***

$$\begin{aligned} \mathbf{x} &= \mathbf{x}^T\mathbf{a}\mathbf{a}^* + \mathbf{x}^T\mathbf{b}\mathbf{b}^* + \mathbf{x}^T\mathbf{c}\mathbf{c}^* = \\ &= \mathbf{x}^T\mathbf{a}^*\mathbf{a} + \mathbf{x}^T\mathbf{b}^*\mathbf{b} + \mathbf{x}^T\mathbf{c}^*\mathbf{c} \end{aligned} \qquad (2)$$

2. INTERSECTION OF 3 PLANES

$$\begin{aligned} \mathbf{x}^T\mathbf{a} = \alpha \ , \ \mathbf{x}^T\mathbf{b} = \beta \ , \ \mathbf{x}^T\mathbf{c} = \gamma \\ \mathbf{x} = \alpha\mathbf{a}^* + \beta\mathbf{b}^* + \gamma\mathbf{c}^* \end{aligned} \qquad (3)$$

3. THE MG-PROBLEM

a′, b′, c′ distances from approximate vectors **x**′ (annex 2, equ. 1a)
a, b, c measured distances (4a)
v_1,v_2,v_3 improvements

$$\begin{aligned} da &= \mathbf{r}_1^T(d\mathbf{x}_s - d\mathbf{x}_1) \\ db &= \mathbf{r}_2^T(d\mathbf{x}_s - d\mathbf{x}_2) \text{ (annex 2, equ. 1b)} \\ dc &= \mathbf{r}_3^T(d\mathbf{x}_s - d\mathbf{x}_3) \end{aligned} \qquad (4b)$$

For the distances a,b,c the following equations exist:

$$\begin{aligned} a' + da &= a + v_1 \\ b' + db &= b + v_2 \\ c' + dc &= c + v_3 \end{aligned} \qquad (5)$$

Introducing equ. (4a,b) a system for $\mathbf{x}_s$ follows

$$\begin{aligned} \mathbf{r}_1^T d\mathbf{x}_s &= \mathbf{r}_1^T d\mathbf{x}_1 + (a - a') + v_1 \\ \mathbf{r}_2^T d\mathbf{x}_s &= \mathbf{r}_2^T d\mathbf{x}_2 + (b - b') + v_2 \\ \mathbf{r}_3^T d\mathbf{x}_s &= \mathbf{r}_3^T d\mathbf{x}_3 + (c - c') + v_3 \end{aligned} \tag{6a}$$

Therefrom an explicit formula for $d\mathbf{x}_s$ can be computed

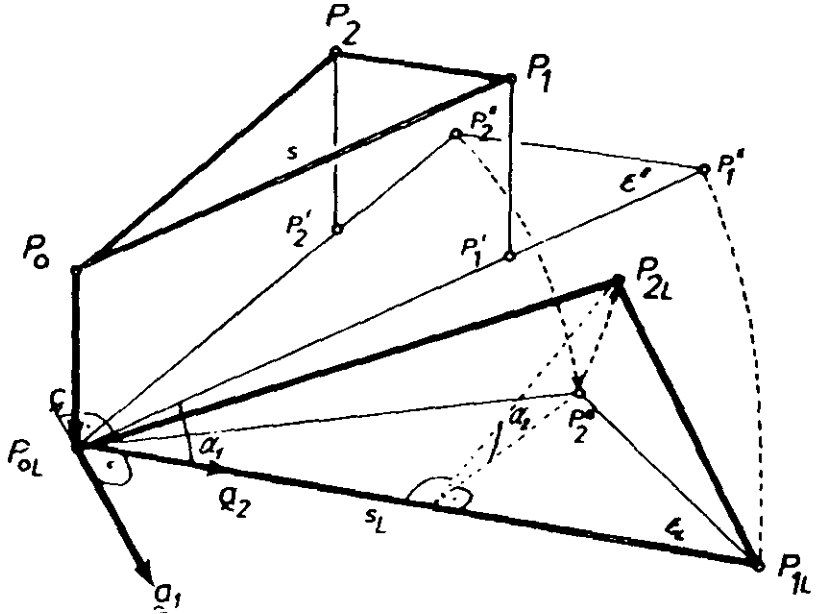
$$d\mathbf{x}_s = \mathbf{v} + \mathbf{l} + \mathbf{W}d\mathbf{x} \tag{6b}$$

$$\begin{aligned} \mathbf{v} &= v_1\mathbf{r}_1^* + v_2\mathbf{r}_2^* + v_3\mathbf{r}_3^* \\ \mathbf{l} &= (a - a')\mathbf{r}_1^* + (b - b')\mathbf{r}_2^* + (c - c')\mathbf{r}_3^* \\ \mathbf{W} &= (\mathbf{W}_1, \mathbf{W}_2, \mathbf{W}_3) , \qquad \mathbf{W}_i = (\mathbf{r}_i^* \mathbf{r}_i^T) \\ d\mathbf{x}^T &= (d\mathbf{x}_1^T, d\mathbf{x}_2^T, d\mathbf{x}_3^T) \end{aligned} \tag{6c}$$

ANNEX 4. 3 D - LINEAR TRANSFORMATION

1. PROBLEM

P(**x**) old system
$P_L(\mathbf{x}_L)$ new system
wanted: linear transformation $\mathbf{x} \Leftrightarrow \mathbf{x}_L$



$$\mathbf{x}_L = \mathbf{c} + \mu\mathbf{R}\mathbf{x} = \mathbf{x}_{oL} + \mu\mathbf{R}(\mathbf{x} - \mathbf{x}_o) \quad (1)$$

$$\mathbf{c} = \mathbf{x}_{oL} - \mu\mathbf{R}\mathbf{x}_o$$

c shift vector

μ scale factor

R orthogonal matrix

The 3D-linear transformation has 7 parameters

$$\mathbf{c}(c_1, c_2, c_3), \mu, \mathbf{R}(\alpha_1, \alpha_2, \alpha_3)$$

For the determination 7 homologous coordinates are necessary:

$$P_0 : \mathbf{x}_0(x, y, z)_0 \rightarrow \mathbf{x}_{0L}(x, y, z)_{0L}$$
$$P_1 : \mathbf{x}_1(x, y, z)_1 \rightarrow \mathbf{x}_{1L}(x, y, z)_{1L}$$
$$P_2 : \mathbf{x}_2(x, y, z)_2 \rightarrow \mathbf{x}_{2L}(-, -, z)_{2L}$$

2. GENERAL SOLUTION

given : $\mathbf{x}_0$, $\mathbf{x}_1$, $\mathbf{x}_2$; $\mathbf{x}_{0L}$, $\mathbf{x}_{1L}$, $\mathbf{x}_{2L}(-, -, z)$
wanted : $\mathbf{c}$, μ , $\mathbf{R}$

2.1. shiftvector

$$\mathbf{c} = \mathbf{x}_{0L} - \mathbf{x}_0 \tag{2a}$$

shifted coordinates $\mathbf{x}_i' = \mathbf{x}_i + \mathbf{c}$

2.2. scalefactor

$$\mu = \frac{|\mathbf{x}_{0L} - \mathbf{x}_{1L}|}{|\mathbf{x}_0 - \mathbf{x}_1|} = \frac{s_L}{s} \tag{2b}$$

scaled coordinates $\mathbf{x}_i'' = \mathbf{x}_{0L} + \mu(\mathbf{x}_i' - \mathbf{x}_{0L})$

2.3. rotation of plane $\varepsilon'' = (\mathbf{P}_{0L}\ \mathbf{P}_1''\ \mathbf{P}_2')$ into plane $\varepsilon_L = (\mathbf{P}_{0L}\ \mathbf{P}_{1L}\ \mathbf{P}_{2L}(z))$ by 2 rotations

a. *rotation around axix* $\mathbf{a}_1$ *till* $P_1'' = P_{1L}$

$\mathbf{a}_1$ is perpendicular to plane $(P_{0L}\ P_1''\ P_{1L})$

$$\mathbf{a}_1 = \frac{(\mathbf{x}_1'' - \mathbf{x}_{0L}) \times (\mathbf{x}_{1L} - \mathbf{x}_{0L})}{|(\mathbf{x}_1'' - \mathbf{x}_{0L}) \times (\mathbf{x}_{1L} - \mathbf{x}_{0L})|}$$

$$\cos\alpha_1 = \frac{(\mathbf{x}_1'' - \mathbf{x}_{0L})^T\ (\mathbf{x}_{1L} - \mathbf{x}_{0L})}{|(\mathbf{x}_1'' - \mathbf{x}_{0L})^T\ (\mathbf{x}_{1L} - \mathbf{x}_{0L})|} \tag{2c,1}$$

axis $\mathbf{a}_1$ and rotation angle α_1 define a rotationmatrix $\mathbf{R}_1$ (see d.)

transformed coordinates $\mathbf{x}_i''' = \mathbf{x}_{0L} + \mathbf{R}_1(\mathbf{x}_i'' - \mathbf{x}_{0L})$

b. *rotation around axis* $\mathbf{a}_2$ *till* $P_2''' = P_{2L}$

$\mathbf{a}_2$ is parallel to line $(P_{0L}\ P_{1L})$

$$\mathbf{a}_2 = \frac{\mathbf{x}_{1L} - \mathbf{x}_{0L}}{|\mathbf{x}_{1L} - \mathbf{x}_{0L}|} = (a_{21},\ a_{22},\ a_{23})^T$$

rotation angle α_2 is given by

$$A\sin\alpha_2 + B\cos\alpha_2 + C = 0$$

$$A = a_{21}(y_2''' - y_{0L}) - a_{22}(x_2''' - x_{0L})$$

$$B = -a_{23}\{a_{21}(x_2''' - x_{0L}) + a_{22}(y_2''' - y_{0L}) + a_{23}(z_2''' - z_{0L})\} + (z_2''' - z_{0L})$$

$$C = a_{23}\{a_{21}(x_2''' - x_{0L}) + a_{22}(y_2''' - y_{0L}) + a_{23}(z_2''' - z_{0L})\} - (z_{2L}''' - z_{0L})$$

with the solutions

$$\tan\frac{\alpha_2}{2} = \frac{1}{C - B}(-A \pm \sqrt{A^2 + B^2 - C^2}) \qquad (2c,2)$$

c. *combined rotation*

$$\mathbf{R} = \mathbf{R}_2\mathbf{R}_1 \qquad (2c,3)$$

d. *matrix R_1 and R_2*

$\mathbf{R} = \mathbf{R}(\mathbf{a},\alpha)$ $\qquad$ $\mathbf{a}^T = (a_1,a_2,a_3)$ rotation axix

α rotation angle

$$\mathbf{R} = (1 - \cos\alpha)\mathbf{a}\mathbf{a}^T + \sin\alpha\mathbf{A}_s + \cos\alpha\mathbf{E} = (a_{ik})$$

$$\begin{array}{ll} a_{11} = (1 - \cos\alpha)a_1^2 + \cos\alpha & a_{12} = (1 - \cos\alpha)a_2a_1 - a_3\sin\alpha \\ a_{21} = (1 - \cos\alpha)a_1a_2 + a_3\sin\alpha & a_{22} = (1 - \cos\alpha)a_2^2 + \cos\alpha \\ a_{31} = (1 - \cos\alpha)a_1a_3 - a_2\sin\alpha & a_{32} = (1 - \cos\alpha)a_3a_2 + a_1\sin\alpha \end{array} \qquad (3)$$

$$a_{13} = (1 - \cos\alpha)a_3a_1 + a_2\sin\alpha$$
$$a_{23} = (1 - \cos\alpha)a_3a_2 - a_1\sin\alpha$$
$$a_{33} = (1 - \cos\alpha)a_3^2 + \cos\alpha$$

2.4. transformation equation

$$\mathbf{x}_{iL} = \mathbf{c} + \mu\mathbf{R}\mathbf{x}_i \qquad (2d)$$

3. LEAST SQUARE ADJUSTMENT

3.1. problem

If $n > 7$ homologous coordinates $\mathbf{x}_i$, $\mathbf{x}_{iL}$ are given, approximate parameters

$$(z) = [(\mathbf{c}),(\mu),(\mathbf{R})]$$

can be calculated from 7 coordinates with the formulas of section 2. By a least square adjustment using all coordinates improved values

$$\mathbf{c} = (\mathbf{c}) + \mathrm{d}\mathbf{c},\ \mu = (\mu),\ \mathbf{R} = \mathrm{d}\mathbf{R}(\mathbf{R})$$

can be calculated.

3.2. formulas

approximate coordinates:

$$(\mathbf{x}_{iL}) = (\mu)(\mathbf{R})\mathbf{x}_i + (\mathbf{c}) \tag{4a}$$

improved coordinates:

$$\bar{\mathbf{x}}_{iL} = (\mathbf{x}_{iL}) + \mathrm{d}\mathbf{c} + \mathrm{d}\mu(\mathbf{x}_{iL}) + (\mathbf{X}_{iL})\mathrm{d}\mathbf{a} \tag{4b}$$

$\mathbf{X}_{iL}$ axiator to vector $\mathbf{x}_{iL}$

The matrix $\mathrm{d}\mathbf{R}$ has to be computed from equ. (3) using the relations

$$\mathrm{d}\alpha_i = \frac{\mathrm{d}\mathbf{a}_i}{\sqrt{\mathrm{d}\mathbf{a}^T\mathrm{d}\mathbf{a}}}\ ,\ \sin\alpha_i = \mathrm{d}\alpha_i\ ,\ \cos\alpha_i = \sqrt{1 - \sin^2\alpha_i}$$

discrepancies:

$$\mathbf{d}_i = (\bar{\mathbf{x}}_L - \mathbf{x}_L)_i = \mathbf{C}_i\mathrm{d}\mathbf{u} + \mathbf{w}_i \tag{4c}$$

$$\mathbf{C}_i = (\mathbf{E}, \mathbf{x}, \mathbf{X})_i = \begin{bmatrix} 1 & o & o & x & o & z & -y \\ o & 1 & o & y & -z & o & x \\ o & o & 1 & z & y & -x & o \end{bmatrix}_{iL}$$

$$\mathrm{d}\mathbf{u}^T = (\mathrm{d}\mathbf{c}^T, \mathrm{d}\mu, \mathrm{d}\mathbf{a}^T)$$
$$\mathbf{w}_i = \{(\mathbf{x}_L) - \mathbf{x}_L\}_i$$

From

$$[\mathbf{d}_i^T\ \mathbf{d}_i]_1^n = \text{Minimum}$$

the normal equations follow:

$$\mathbf{C}^T\mathbf{C}\mathrm{d}\mathbf{u} + \mathbf{C}^T\mathbf{w} = \mathbf{O} \tag{5}$$

$$\mathbf{C}^T = (\mathbf{C}_1^T, \mathbf{C}_2^T \ldots\ldots \mathbf{C}_n^T)$$

$$\mathbf{w}^T = (\mathbf{w}_1^T, \mathbf{w}_2^T \ldots\ldots \mathbf{w}_n^T)$$

For coordinates $\mathbf{x}'$ related to the center of mass

$$\mathbf{x}' = \mathbf{x} - \mathbf{x}_s,\ \mathbf{x}_s = \frac{1}{n}[\mathbf{x}]\ ,\ [\mathbf{x}'] = \mathrm{O} \tag{6a}$$

explicit solutions for the shiftvector and scalefactor exist:

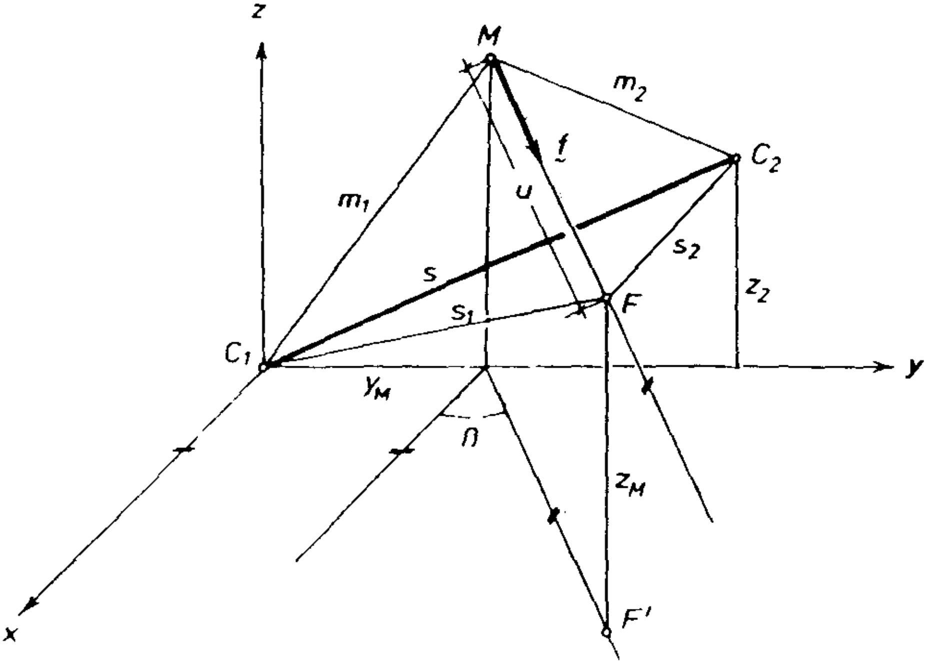
$$\mathbf{dc} = \frac{1}{n}[\mathbf{w}]$$
$$d\mu = \frac{[(\mathbf{x}_L')^T\mathbf{w}]}{[(\mathbf{x}_L')^2]} \tag{6b}$$

The components of $\mathbf{da}$ can be calculated from the following equations:

$$\begin{bmatrix} [y'^2 + z'^2] & -[x'y'] & -[z'x'] \\ -[x'y'] & [x'^2 + z'^2] & -[y'z'] \\ -[z'x'] & -[y'z'] & [x'^2 + y'^2] \end{bmatrix} \begin{bmatrix} da_1 \\ da_2 \\ da_3 \end{bmatrix}_L + \begin{bmatrix} [y'w_z - z'w_y] \\ [z'w_x - x'w_z] \\ [x'w_y - y'w_x] \end{bmatrix}_L = \mathbf{O} \tag{6c}$$

ANNEX 5. THEORY OF LINE CROSSING

The length of a distance $s = C_1 C_2$, the terminals C_1, C_2 of which are lying in different heights z_1, z_2, should be determined by the procedure of line crossing. The crossing line should take a horizontal course **f**, with the altitude z_M, and enclose an angle of $90^\circ + \beta$ with the vertical plane through C_1 and C_2. From points C_1, C_2 will be measured.
Required is the distance u_m of the point F_m, having a minimum sum $S = s_1 + s_s$, from the crossing point M where the crossing line penetrates through the vertical plane.



The two distances belonging to point M should be called m_1, m_2 and the differences of altitude of this point and point C_2 opposite to C_1 with z_M and z_2.
For further investigations a coordinatesystem through C_1 will be selected, with the z-axis parallel to the perpendicular and the y-axis of this system being part of the vertical plane through C_1 and C_2.
In this system the following relations for the position vector of the points C_1, C_2, M and F are valid:

$$\mathbf{x}_1^T = (O, O, O) \qquad \mathbf{x}_2^T = (O, y_2, z_2)$$

$$\mathbf{x}_M^T = (O, y_M, z_M) \qquad \mathbf{x}_F = \mathbf{x}_M + u\mathbf{f}$$

$$\mathbf{f} = (\cos\beta, \sin\beta, O) \qquad (1)$$

With these relations the distances s_1, s_2 may be expressed as a function of the parameter u, which equals the required distance between the points F and M.

$$\begin{aligned} s_1^2 &= x_F^2 = u^2 + m_1^2 + 2uy_M \sin\beta \\ s_2^2 &= (x_F - x_2)^2 = u^2 + m_2^2 + 2u(y_M - y_2)\sin\beta \end{aligned} \tag{2}$$

The determination of u_m demands a minimum sum of distances $S(u) = s_1+s_2$ leading to the equation:

$$\frac{dS}{du} = \frac{1}{s_1}(u + y_M \sin\beta) + \frac{1}{s_2}(u + (y_M - y_2)\sin\beta) = O \tag{3}$$

After several transformations as result a quadratic equation for u_m follows:

$$\begin{aligned} &Au_m^2 + Bu_m + C = O \\ &A = (m_2^2 - m_1^2) + y_2(2y_M + y_2)\sin\beta \\ &B = \{y_M(m_2^2 - m_1^2) + y_2m_1^2 + y_My_2(y_M - y_2)\}\sin^2\beta \\ &C = \{y_M^2(m_2^2 - m_1^2) + y_2(2y_M - y_2)m_1^2\}\sin^2\beta \end{aligned} \tag{4}$$

The discussion of equation (4) gives the following items:

a. For $\beta=O$, hence, a horizontal crossing line normal to $s=C_1C_2$, resulting in $u_m = O$ as a solution of the equation. In this case the crossing point M coincides with point F_m, with its minimum sum of distances independent of the position of the crossing point M.
b. Independent of the value of the crossing angle, the value of u_m will likewise be zero, if the points C_1,C_2 have the same altitude $z_1=z_2$ and the crossing point lies in the middle of the straight line s that means $y_2 = 2y_M$.
c. For low values of the distance u, the following relation is valid

$$u_m = \frac{1}{2} \frac{y_M^2(m_2^2-m_1^2) + y_2(2y_M-y_2)m_1^2}{y_M(m_2^2-m_1^2) + y_2 m_1^2 + y_M y_2 (y_M-y_2) \sin\beta} \tag{5}$$

Example

For the configuration

	C_1	C_2	M
x	0	0	0
y	0	10 000	7 000
z	0	80	1 300

(x,y,z in meter)

and the crossing angles $\beta_1 = 5^g$, $\beta_2 = 30^g$ equation (5) yields the values $u_{m_1} = 9$ m and $u_{m_2} = 90$ m respectively.

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