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A formalisation of the "step forward-step backward" reasoning

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Resumen

Parece que nuestro razonamiento cotidiano consiste en dos pasos. Uno "adelante" extendiendo nuestras creencias, el segundo "atrás" reduciéndolas. La lógica deductiva formaliza el primero, pero las lógicas que formalizan el rechazo de sentencias no sirven aquí. Necesitamos dos lógicas: una trabajando con conjuntos de sentencias aceptadas y otra con sentencias rechazadas. Dado que trabajamos con la misma clase de conjuntos, el segundo es un conjunto de sentencias aceptadas que es reducido por el razonamiento.

Palabras clave: Lógica, razonamiento cotidiano, lógica deductiva, lógica no-monótona, consecuencia lógica.

Abstract

Our everyday thinking consists of two steps: "forward" extending our beliefs, "backward" reducing them. The "forward" step is formalized by deductive logic, but existing logics formalising "rejected sentences" reasoning are unvalid for the "backward" reasoning. We need two logics: one for the set of accepted sentences, another for the set of rejected sentences. They work on the same class of sets, so the second component of the pair must be a reasoning decreasing sets of accepted sets.

Keywords: Logic, everyday reasoning, deductive logic, nonmonotonic logic, logical consequence.

1. The reductive classical elimination operation

The logic inferring true sentences from the set of true sentences will be here called the logic of truth. Analogously, the logic inferring false sentences from the set of false sentences will be called the logic of falsehood. It seems that from the syntactical point of view it is impossible to talk on the logic of truth as well as on the logic of falsehood. However, even syntactical approach enables the relative distinction between both logics. If a given logic will be understood as a logic of truth, then the logic dual to it should be interpreted as the logic of falsehood and vice versa.

Let the language for the classical logic is an algebra

$$\mathcal{L} = (\mathrm{L},\neg,\wedge,\vee,\rightarrow)$$

Let assume that X is a set of accepted sentences. Then L-X is a set of rejected sentences. If C' is a function dual to the classical consequence operation C, then C'(L-X) is a set of all these sentences which should be understood as false because of falsehood of the sentences from L-X. Although L-X $\subseteq$ C'(L-X), L-C'(L-X) $\subseteq$ L-(L-X) = X. It means that using a logic of falsehood it is possible to reduce the set of accepted sentences. Indeed, our given at the beginning set X should be decreased by all sentences from C'(L-X). Thus, our first problem it to reconstruct the function dual to the classical consequence operation.

In [7] Wójcicki defined C^d an operation dual to the given consequence operation as follows:

$$\alpha \in C^d(X) \textit{ iff } \cap\{C(\beta)\colon \beta\in X_f\} \subseteq C(\alpha), \textit{for some finite } X_f \subseteq X$$

for any X$\subseteq$L. Of course, C^d is a finitary and structural consequence operation. Moreover, C^d can be semantically defined by the matrix $((\{0,1\},-,\cap,\cup,\Rightarrow),\{0\})$, with the well known classical conditions interpreting negation, conjunction, disjunction and implication. Let H be a class of classical valuations h: L $\rightarrow$ {0,1}. Then,

$$\alpha \in C(X) \textit{ iff } \forall\eta\in H\ (h(X) \subseteq\{1\} \text{ implies } h(\alpha) = 1)$$
$$\alpha \in C^d(X) \textit{ iff } \forall\eta\in H\ (h(X) \subseteq\{0\} \text{ implies } h(\alpha) = 0)$$

Operations C and C^d have axiomatizations expressing one and the same thinking. For the better comparison let us recall the well known axiom set for

C:

1^{+D} $\varnothing$ ē $\alpha \to (\beta \to \alpha)$
2^{+D} $\varnothing$ ē $(\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$
3^{+D} $\varnothing$ ē $(\alpha \wedge \beta) \to \alpha$
4^{+D} $\varnothing$ ē $(\alpha \wedge \beta) \to \beta$
5^{+D} $\varnothing$ ē $(\alpha \to \beta) \to ((\alpha \to \gamma) \to (\alpha \to (\beta \wedge \gamma)))$
6^{+D} $\varnothing$ ē $\alpha \to (\alpha \vee \beta)$
7^{+D} $\varnothing$ ē $\beta \to (\alpha \vee \beta)$
8^{+D} $\varnothing$ ē $(\alpha \to \gamma) \to ((\beta \to \gamma) \to ((\alpha \vee \beta) \to \gamma))$
9^{+D} $\varnothing$ ē $(\alpha \to \neg\beta) \to (\beta \to \neg\alpha)$
10^{+D} $\varnothing$ ē $\neg(\alpha \to \alpha) \to \beta$
11^{+D} $\varnothing$ ē $\alpha \vee \neg\alpha$
MP^{+D} $\{\alpha, \alpha \to \beta\}$ ē β

The syntax of C^d is the following:

1^{-D} $\varnothing$ ē $\neg(\neg(\alpha \to \beta) \to \alpha)$
2^{-D} $\varnothing$ ē $\neg(\neg(\neg(\gamma \to \alpha) \to \neg(\beta \to \alpha)) \to \neg(\neg(\gamma \to \beta) \to \alpha))$
3^{-D} $\varnothing$ ē $\neg((\alpha \wedge \beta) \to \alpha)$
4^{-D} $\varnothing$ ē $\neg((\alpha \wedge \beta) \to \beta)$
5^{-D} $\varnothing$ ē $\neg(\neg(\neg(\gamma \to (\alpha \wedge \beta)) \to \neg(\gamma \to \beta)) \to \neg(\gamma \to \alpha))$
6^{-D} $\varnothing$ ē $\neg(\alpha \to (\alpha \vee \beta))$
7^{-D} $\varnothing$ ē $\neg(\beta \to (\alpha \vee \beta))$
8^{-D} $\varnothing$ ē $\neg(\neg(\neg((\alpha \vee \beta) \to \gamma) \to \neg(\beta \to \gamma)) \to \neg(\alpha \to \gamma))$
9^{-D} $\varnothing$ ē $\neg(\neg(\alpha \to \neg\beta) \to \neg(\beta \to \neg\alpha))$
10^{-D} $\varnothing$ ē $\neg(\beta \to (\alpha \to \alpha))$
11^{-D} $\varnothing$ ē $\alpha \wedge \neg\alpha$
MT^{-D} $\{\beta, \neg(\alpha \to \beta)\}$ ē α

At first let us notice that axioms for C^d are written in the "opposite" direction in comparison with axioms for C. Thanks to this difference both operations can work on the same sets of sentences. The form of the rules MP^{+D} and MT^{-D} enables to use them for the same implication $\alpha \to \beta$. Indeed, if $\neg(\alpha \to \beta)$ is false, then $\alpha \to \beta$ is true, and so the truth of α means the truth of β as well as the falsehood of β means the falsehood of α. Remembering on the fact of the mutually opposite transcript of formulas from axioms for C and for C^d let us successively compare all of them.

1^{+D} and 1^{-D} on the base of two-valued logic state an obvious fact, that if

α has a given logic value, true in the case of C and false for C^d, whatever other circumstances (expressed here by β), α always has the same value.

2^{+D} can be understood as follows: if the truth of α implies that the truth of β implies the truth of γ, then if the truth of α implies the truth of α, then the truth of α implies the truth of γ. However, it is sufficient to replace the word "truth" by "falsehood" and such obtained sentence will interpret 2^{-D}.

It is easy to see that next both six axioms define the same connectives of conjunction and disjunction. Obviously, in comparison with 3^{+D}-8^{+D}, the conjunction is replaced by the disjunction and the disjunction by the conjunction in 3^{-D}-8^{-D}, which results from the duality of both connectives.

In the case of 9^{+D} and 9^{-D} the situation is especially clear. Both axioms express the same thought: if the truth of the first sentence implies that the second sentence is false, then the truth of the second sentence implies the falsehood of the first sentence.

10^{+D} and 10^{-D} are obviously dual. The first one says that if we accept some absurd sentence, then we have to accept every sentence. According to the second axiom: if we reject some obvious sentence, we have to reject every sentence. Thus both axioms taken together order us to esteem logic expressed by its set of tautologies and the set of counter-tautologies.

Similarly, 11^{+D} and 11^{-D} have a dual character. On the ground of the two-valued logic both axioms together say that if one sentence is a negation of the other one, then exactly one of them is true (is false).

Thus, it seems that both axiomatizations express the same logic: the first syntax from the point of view of the truth, while the second syntax from the point of view of the falsehood.

After the reconstruction of the classical logic of falsehood we can define the operation of the classical elimination E.

For any $X \subseteq L$:

$$E(X) = L\text{-}C^d(L\text{-}X)$$

Let us notice that such defined elimination operation E is unique for C. Indeed, using a dual to the Wójcicki's definition one can define E^d:

$$\alpha \in Ed(X) \textit{ iff } E(L\text{-}\alpha) \subseteq \cup\{E(L\text{-}\beta)\colon \beta \in X_{cf}\}, \text{ for some co-finite } X \subseteq X_{cf}$$

for any $\alpha \in L$ and $X \subseteq L$. Then, the next definition

$$C(X) = L\text{-}\ E^d(L\text{-}X)$$

closes the circle.

Both elimination operations E and E^d satisfy conditions dual to the well known Tarski's conditions for the consequence operation ([6]):

$E(X) \subseteq X$
$X \subseteq Y$ *implies* $E(X) \subseteq E(Y)$
$E(X) \subseteq EE(X)$

Thus every elimination operation is monotonic. Moreover, the classical elimination operation is co-finitary and structural, if respectively:

$E(X) = \cap\{E(Y)$: $X \subseteq Y$ *and* Y *is a co-finite set*$\}$
$e(L\text{-}E(X)) \subseteq L\text{-}E(L\text{-}e(L\text{-}X))$, *for any endomorphism e of the language L*

Naturally, differences between deductive and reductive parts of the logic should be appropriately expressed by their axiomatization. Indeed, as an axiom of deduction (D-axiom) says which formula has to be accepted even when nothing has been accepted at the beginning, thus an axiom of reduction (R-axiom) should informed which formula has to be rejected even when nothing has been rejected at the beginning. Thus, axioms of deductive and reductive part of logic have respectively the following form:

$\varnothing$ ē α and L -| α

The case of rules is similar. The deduction rule (D-rule) is of the form $\varnothing$ + $\{\alpha_1,...,\alpha_\kappa\}$ ē β, in short

$\{\alpha_1,...,\alpha_\kappa\}$ ē β

the reductive rules (R-rules) should be of the shape

L - $\{\alpha_1,...,\alpha_\kappa\}$ ē -| β

The sense of the R-rule is intuitive: a removing of sentences $\{\alpha_1,...,\alpha_\kappa\}$ from any set removes from this set. New shape of axioms and rules for the elimination operation means that it is necessary to formulate a notion dual to the notion of proof:

Let $A_{-|}$ be an axiom set for elimination and $R_{-|}$ be a set of rules of elimination. The formula is called to be disprovable for X *by means of rules from $R_{-|}$, if and only if there exists in* L-X *a finite sequence of formulas* $\alpha_1,...,\alpha_K$, *called a disproof of for* X *by means of $R_{-|}$, such that*

1. $\alpha = \alpha_k$ *and*

2. for any for any $i \in \{1,...,k\}$, $\alpha i \in A\text{-}| \cup (L\text{-}X)$ *or for some* $Y \subseteq \{\alpha_1,...,\alpha_{k-1}\}$, L-Y -| α_i *is an instance of some rule from $R_{-|}$.*

A formula is called to be confirmed for X *by means of $R_{-|}$, if in* L-X *there exists no disproof of for* X *by means of $R_{-|}$.*

Thus, the reductive part of the classical logic of truth E is given by the following R-axioms and R-rule:

1^{+R} L -| $\neg(\neg(\alpha \rightarrow \beta) \rightarrow \alpha)$
2^{+R} L -| $\neg(\neg(\neg(\gamma \rightarrow \alpha) \rightarrow \neg(\beta \rightarrow \alpha)) \rightarrow \neg(\neg(\gamma \rightarrow \beta) \rightarrow \alpha))$
3^{+R} L -| $\neg((\alpha \wedge \beta) \rightarrow \alpha)$
4^{+R} L -| $\neg((\alpha \wedge \beta) \rightarrow \beta)$
5^{+R} L -| $\neg(\neg(\neg(\gamma \rightarrow (\alpha \wedge \beta)) \rightarrow \neg(\gamma \rightarrow \beta)) \rightarrow \neg(\gamma \rightarrow \alpha))$
6^{+R} L -| $\neg(\alpha \rightarrow (\alpha \vee \beta))$
7^{+R} L -| $\neg(\beta \rightarrow (\alpha \vee \beta))$
8^{+R} L -| $\neg(\neg(\neg((\alpha \vee \beta) \rightarrow \gamma) \rightarrow \neg(\beta \rightarrow \gamma)) \rightarrow \neg(\alpha \rightarrow \gamma))$
9^{+R} L -| $\neg(\neg(\alpha \rightarrow \neg\beta) \rightarrow \neg(\beta \rightarrow \neg\alpha))$
10^{+R}L -| $\neg(\beta \rightarrow (\alpha \rightarrow \alpha))$
11^{+R}L -| $\alpha \wedge \neg\alpha$
MT^{+R} L - $\{\beta, \neg(\alpha \rightarrow \beta)\}$ -| α

Especially easy task is to formulate an axiomatization of the reductive part of the classical logic of falsehood E^d:

1^{-R} L -| $\alpha \rightarrow (\beta \rightarrow \alpha)$
2^{-R} L -| $(\alpha \rightarrow (\beta \rightarrow \gamma)) \rightarrow ((\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma))$
3^{-R} L -| $(\alpha \wedge \beta) \rightarrow \alpha$
4^{-R} L -| $(\alpha \wedge \beta) \rightarrow \beta$
5^{-R} L -| $(\alpha \rightarrow \beta) \rightarrow ((\alpha \rightarrow \gamma) \rightarrow (\alpha \rightarrow (\beta \wedge \gamma)))$
6^{-R} L -| $\alpha \rightarrow (\alpha \vee \beta)$
7^{-R} L -| $\beta \rightarrow (\alpha \vee \beta)$
8^{-R} L -| $(\alpha \rightarrow \gamma) \rightarrow ((\beta \rightarrow \gamma) \rightarrow ((\alpha \vee \beta) \rightarrow \gamma))$

9^{-R} L -| $(\alpha \to \neg\beta) \to (\beta \to \neg\alpha)$
10^{-R} L -| $\neg(\alpha \to \alpha) \to \beta$
11^{-R} L -| $\alpha \vee \neg\alpha$
MP^{-R} L – $\{\alpha, \alpha \to \beta\}$ -| β

Analogously to the deduction theorem we have here the reduction theorem. For E it is of the following form:

$\neg(\alpha\to\beta) \in E(X)$ *iff* $\alpha \in E(X\text{-}\beta)$

and for E^d:

$(\alpha\to\beta) \in E^d(X)$ *iff* $\alpha \in E^d(X\text{-}\beta)$

From now, a logic in its *complete* form will be a pair of triples:

(L,C,E) *and* (L,C^d,E^d)

where the first triple is the deductive-reductive logic of truth and enables us to exam the set of true sentences: deductively and reductively; checking if X contains all sentences following as true from X, and if X does not contain sentences following as false from L-X. The second triple is the deductive-reductive logic of falsehood, and enables us a similar deductive and reductive examination of the set of false sentences.

A set X such that X=C(X) ($X=C^d(X)$) will be called a *D-theory* (a *D^d-theory*). Similarly, every X such that X=E(X) ($X=E^d(X)$) will be called a *R-theory* (a *R^d-theory*). If R-theory (R^d-theory) $F \neq \varnothing$, then F is called *sufficient*. Obviously, $E(X) \subseteq X \subseteq C(X)$ and $E^d(X) \subseteq X \subseteq C^d(X)$, for any $X \subseteq L$. It is not difficult to check that there are such sets X, for which inclusions can be replaced by equalities in the expressions above. Let us notice that the following proof of this fact holds also for the intuitionistic logic (see [5]).

For every D-theory T:

$\alpha \in T$ *and* $\beta \in T$ *iff* $\alpha \wedge \beta \in T$

and

$\alpha \in T$ *or* $\beta \in T$ *implies* $\alpha \vee \beta \in T$

and for every R-theory F:

$\alpha \in F$ *and* $\beta \in F$ *if* $\alpha \wedge \beta \in F$

and

$\alpha \in F$ *or* $\beta \in F$ *iff* $\alpha \vee \beta \in F$

Thus, D-theory T and R-theory F are prime if respectively:

$\alpha \in T$ *or* $\beta \in T$ *if* $\alpha \vee \beta \in T$

and

$\alpha \in F$ *and* $\beta \in F$ *implies* $\alpha \wedge \beta \in F$

Prime R-theories of the classical, and also intuitionistic logic exist because there exist relatively minimal R-theories. Let us recall that T is relatively maximal D-theory, if it is maximal relatively to some δ, i.e. if for some $\delta \notin T$: $\delta \in C(T \cup \{\beta\})$, for any $\beta \notin T$. Thus, F is a minimal relatively R-theory, if F is minimal relatively to some δ, i.e. if for some $\delta \in F$: $\delta \notin E(F-\{\beta\})$, for any $\beta \in F$.

Assume that F is a R-theory minimal relatively to δ. Let moreover, $\alpha \in F$ and $\beta \in F$. Then, $\delta \notin E(F-\{\alpha\})$ and $\delta \notin E(F-\{\beta\})$. So, by reduction theorem, $\neg(\delta \to \alpha) \notin F$ $\alpha v \delta$ $\neg(\delta \to \beta) \notin F$. By 5^{+R} and MT^{+R}, $\neg(\delta \to (\alpha \wedge \beta)) \notin F$. Since $\delta \in F$, thus $\alpha \wedge \beta \in F$. It means that F is prime R-theory.

Let F be a prime R-theory such that $\alpha \in F$ and $\alpha \to \beta \in F$. Since $(\alpha \to \beta) \wedge \neg(\alpha \to \beta) \notin F$, so $\neg(\alpha \to \beta) \notin F$. Thus, $\beta \notin F$ implies $\alpha \notin F$, by MT^{+R}. By assumption $\beta \in F$, and so F is a D-theory. It proves that every prime R-theory is a prime D-theory. It is also easy to show that every prime D-theory is a prime R-theory. Similarly, every prime R^d-theory (D^d-theory) is a prime D^d-theory (R^d-theory). Thus maximal classical D-theories and D^d-theories are closed also on the elimination operations E and E^d, respectively.

The above notion of the relatively maximality and minimality can be easily extended to the following:

Let C *be a consequence operation on L,* $\alpha_1,\ldots,\alpha_k \in L$ *and* $T \subseteq L$. T *is a D-theory maximal relatively to the formula set* $\{\alpha_1,\ldots,\alpha_k\}$, *if there are satisfied two following conditions:*

(a) $\alpha_1 \notin T$, *for any* $i \in \{_1,\ldots,_k\}$.

(b) $\alpha_i \in C(T+\beta)$ *for some* $i \in \{_1,...,_k\}$, *provided* $\beta \notin T$.

Let E *be a consequence operation on L,* $\alpha_1,...,\alpha_k \in L$ *and* $F \subseteq L$. F *is a R-theory maximal relatively to the formula set* $\{\alpha_1,...,\alpha_k\}$, *if there are satisfied two following conditions:*

(a) $\alpha_1 \in F$, *for any* $i \in \{_1,...,_k\}$.

(b) $\alpha_i \notin E(F\text{-}\beta)$ *for some* $i \in \{_1,...,_k\}$, *provided* $\beta \notin T$.

Of course, a D-theory maximal relatively to the set $\{\alpha_1,...,\alpha_k\}$ is maximal relatively to $\alpha_1 \vee ... \vee \alpha_K$ and vice versa. Thus, a D-theory T maximal relatively to is maximal relatively to the set $\{\alpha\} \cup Y$, for any Y such that $Y \cap T = \emptyset$; but not conversely. Similarly, a R-theory minimal relatively to the set $\{\alpha_1,...,\alpha_k\}$ is minimal relatively to $\alpha_1 \vee ... \vee \alpha_K$ and vice versa. A R-theory F minimal relatively to a is minimal relatively to the set $\{\alpha\} \cup Y$, for any Y such that $Y \subseteq T$; but not conversely.

An analogous to the deductive Lindenbaum lemma is a reductive dual-to-Lindenbaum lemma:

Let E *be a co-finitary elimination operation on L. For any sufficient R-theory* T *and for any* $\alpha \in T$ *there exists a R-theory* T_0 *minimal relatively to such that* $T_0 \subseteq T$.

A procedure decreasing sufficient R-theory to the R-theory minimal relatively to some formula is analogous to the construction, known from the proof of the Lindenbaum lemma (e.g. [8]), extending consistent D-theory to the D-theory maximal relatively to some formula.

Naturally, both lemmas can be also extended to the more general case with the set $\{\alpha_1,...,\alpha_k\}$ instead of α.

Directly from the proofs of both lemmas, it is easy to see that for any consistent D-theory T and for any formula $\alpha \notin T$, there exist infinitely many D-theories including T and maximal relatively to ; as well as for any sufficient R-theory F and for any formula $\alpha \in F$, there exist infinitely many R-theories included in F and minimal relatively to α. It means that, "a C-theory maximal relatively to" and "a R-theory minimal relatively to α" name infinite classes of objects. Let us distinguish some subclasses of classes mentioned above.

Let C *and* E *be, respectively, a consequence and elimination operations on L,* $\alpha \in L$, $X,Y \subseteq L$.

1. A D-theory T *maximal relatively to is for the set* X, *if* T *contains a maximal amount of formulas from* X;

2. A R-theory F *minimal relatively to is against the set* X, *if* F *contains a minimal amount of formulas from* X.

Let C *and* E *be, respectively, a consequence and elimination operations of the classical propositional logic on L,* $\alpha \in L$, $X,Y \subseteq L$.

3. A D-theory T *maximal relatively to is against the set* X, *if* T *contains a minimal amount of formulas from* X;

4. A R-theory F *minimal relatively to is for the set* X, *if* F *contains a maximal amount of formulas from* X.

Ad. 1,2: In order to construct a maximal relatively to α D-theory for the set X including consistent D-theory T such that $\alpha \notin T$, it is sufficient to modify a proof of Lindenbaum lemma, presented with all details in [8]. This construction begins from the ordering, in the form of the sequence, of all formulas of the set L-T. Building a maximal relatively to α D-theory for the set X, this sequence is formed in such a way that every formula from X proceeds all formulas from (L-T)-X. It means that all formulas from the set X will be always taken into account as first in procedure of extending of the D-theory T.

An analogous, small correction of the proof of the dual-to-Lindenbaum lemma results in the construction of minimal relatively R-theories against a given set X. For a given sufficient R-theory F, the initial sequence $\alpha_1,\alpha_2,...,\beta_1,\beta_2,...$ is such that $a_i \in X$ and $b_j \in F$-X for any i,j.

Ad. 3,4: An appropriate construction for the relatively maximal D-theory against a given set X is the repetition of the construction presented above with the difference that in the initial sequence of formulas from L-T, every formula from $X^{\neg}$ proceeds all formulas from (L-T)-$X^{\neg}$, where $X^{\neg} = \{\neg\alpha; \alpha \in X\}$. In such a way, all formulas from the set $X^{\neg}$ will be always first taking into account in the procedure of extending of the D-theory T. Since every obtained D-theory is consistent, it is impossible to add some formula $\alpha \in X$ during a construction, if $\neg\alpha$ has already been added.

Similarly, for a given sufficient R-theory F, a construction of the relatively minimal R-theory included in F for the set X uses the initial sequence of formulas from F such that every formula from $X^{\neg}$ proceeds all formulas from F-$X^{\neg}$ in this sequence. It means that at first we will remove from F all these formulas which belong to the set $X^{\neg}$. Since, for any $\alpha \in L$, one formula from the pair $(\alpha,\neg\alpha)$ has to belong to every R-theory, it is impossible that some formula $\alpha\in X$ will be removed during a construction, if $\neg\alpha$ has already been removed.

Let us notice that in the proof of the point 3 the set L-X cannot replaced the set $X^{\neg}$. Indeed, let $X = \{p\to q\}$ and T be any consistent D-theory which does not contain any formula with p, q or with t. Of course, $X^{\neg} = \{p\wedge\neg q\}$. If formula from $X^{\neg}$ proceeds all formulas from (L-T)- $X^{\neg}$, then the maximal relatively to t D-theory T_0 against the set X does not contain $p\to q$. Now assume that L-X plays the role of $X^{\neg}$, i.e. every formula from L-X proceeds all formulas from (L-T)-(L-X). Then, if $q\notin X$ will be the first formula in the sequence of all formulas from L-T, then $p\to q$ will belong to the maximal relatively to t D-theory T_0 against the set X. Then, a D-theory aginst the set X become de facto a D-theory for the set X. Similarly, the set $X^{\neg}$ cannot be replaced by the set L-X in the proof of 4. This is a reason for the restriction of 3 and 4 to the case of the logic with the classical negation.

Semantics for C, E, C^d and E^d is defined by one class of CL-models for the classical propositional logic. If $A = (A,-,\cap,\cup,\Rightarrow)$ is an algebra similar to the language L and D is a non-empty subset of A, then the matrix $M = (A,D)$ is a CL-model, if

(-) $-a\in D$	*iff*	$a\notin D$
$(\cap)$ $a\cap b\in D$	*iff*	$a\in D$ *and* $b\in D$
$(\cup)$ $a\cup b\in D$	*iff*	$a\in D$ *or* $b\in D$
$(\to)$ $a\to b\in D$	*iff*	$-a\in D$ *or* $b\in D$

for any \$$a,b\in A$. CL-model M is a base for defining four operations C_M, C^d_M, E_M and E^d_M.

$\alpha \in C_M(X)$ *iff* $\forall h\in Hom(L,A)$ $(\forall\beta\in X\ h(\beta)\in D$ *implies* $h(\alpha)\in D)$
$\alpha \in C^d_M(X)$ *iff* $\forall h\in Hom(L,A)$ $(\forall\beta\in X\ h(\beta)\notin D$ *implies* $h(\alpha)\notin D)$
$\alpha \in E_M(X)$ *iff* $\exists h\in Hom(L,A)$ $(\forall\beta\notin X\ h(\beta)\notin D$ *and* $h(\alpha)\in D)$
$\alpha \in E^d_M(X)$ *iff* $\exists h\in Hom(L,A)$ $(\forall\beta\notin X\ h(\beta)\in D$ *and* $h(\alpha)\notin D)$

Let M be a class of all CL-models. Then, the matrix consequence operation, the matrix dual consequence operation, the matrix elimination operation and the matrix dual elimination operation are following:

$\alpha \in C_M(X)$ *iff* $\alpha \in C_M(X)$, *for every CL-model M*

$\alpha \in C^d{}_M(X)$ *iff* $\alpha \in C^d{}_{\mathcal{M}}(X)$, *for every CL-model M*
$\alpha \in E_M(X)$ *iff* $\alpha \in E_{\mathcal{M}}(X)$, *for some CL-model M*
$\alpha \in E^d{}_M(X)$ *iff* $\alpha \in E^d{}_{\mathcal{M}}(X)$, *for some CL-model M*

Thus,

$\alpha \notin E_M(X)$ *iff* $\alpha \in C^d{}_M(L\text{-}X)$
$\alpha \in E^d{}_M(X)$ *iff* $\alpha \in C_M(L\text{-}X)$

Obviously, completeness theorems are following:

$\alpha \in C(X)$ *iff* $\alpha \in C_M(X)$
$\alpha \in C^d(X)$ *iff* $\alpha \in C^d{}_M(X)$
$\alpha \in E(X)$ *iff* $\alpha \in E_M(X)$
$\alpha \in E^d(X)$ *iff* $\alpha \in E^d{}_M(X)$

2. "Step forward - step backward" reasoning

As it was already mentioned, a logic in its complete form, i.e. a deductive-reductive logic of truth together with a deductive-reductive logic of falsehood enables to formalise a non-monotonic reasoning. Indeed, let assume that today we have an appointment in the restaurant with somebody. If we get a new premise "it is cold today" behind of the fact of the meeting we know that we have to take an overcoat. If we get another information "it is raining today" we additionally know that we have to take an umbrella. Up to now, we were making only steps forward permanently expanding our set of conclusions. Thus, an extension of the set of premises extends the set of conclusions. It means that we could twice use a consequence operation.

However, it is sufficient to assume that our next new premise is an information unsaying the meeting. Then, we have to make a step backward, i.e. to cancel some (maybe all?) so far obtained conclusions. It appear that a formalisation of this step is possible thanks to the elimination operation. Thus, a logic given in the deductive-reductive form can be a tool for the formal expression of the "step forward - step backward" reasoning.

For the systematic presentation let us begin from the *expansion of truth* being a closure of the set of true sentences on the consequence operation C. Let assume that a D-theory T^+ is our set of beliefs. Let moreover, $A_1,...,A_n$ be our new beliefs. Then, our new set of beliefs is again a D-theory and it has the following form:

$$T^+ \oplus \{A_1,...,A_n\} = C(T^+ \cup \{A_1,...,A_n\})$$

Usually, our opinions consist also of beliefs expressing negative facts. Sentences representing these beliefs are for us false sentences, e.g. *"dwarfs exist"*. Naturally, we infer another false sentences from these beliefs: *"we can meet dwarf"*, *"we can expect a dwarf's help"* etc. A logic in complete form enables us to formalise also this reasoning, an *expansion of falsehood* being a closure on the consequence operation C^d. Now, assume that D^d-theory T- is a set of our negative beliefs. Then, if we add next false sentences $A_1,...,A_n$ our new negative belief set is as follows:

$$T^- \oplus \{A_1,...,A_n\} = C^d(T^- \cup \{A_1,...,A_n\})$$

Both expansions represent steps forward in the reasoning. In the first case it is developed the set of true sentences, while in the second case the set of false sentences. Now, let us consider a formalisation of the thinking reducing the set of beliefs.

As previously, let D-theory T^+ be a set of our beliefs. Assume that because of some reason sentences $A_1,...,A_n$ became for us already false. It means that we have to reduce T^+ by these sentences. Thus, we should make a step backward in the reasoning i.e. a *contraction of truth* of the first kind:

$$(T^+ \ominus \{A_1,...,A_n\})^1{}_{B1,...,Bk} = T^+ \cap E^{T+}{}_{B1,...,Bk}(L\text{-}\{A_1,...,A_n\})$$

or maybe of the second kind:

$$(T^+ \ominus \{A_1,...,A_n\})^2{}_{B1,...,Bk} = T^+ \cap nE^{T+}{}_{B1,...,Bk}(L\text{-}\{A_1,...,A_n\})$$

$E^{T+}{}_{B1,...,Bk}(L\text{-}\{A_1,...,A_n\})$ is a R-theory for T^+ minimal relatively to the set $B_1,...,B_k$. Assume that $A_1 \in T$ follows from $Z=\{C_1,...,C_s\}$, and A_1 does not follow from any proper subset of Z. A rejection of A_1 from T removes a disjunction $C_1 \vee ... \vee C_s$. Unfortunately, all sentences from Z still belong to $E(L\text{-}\{A_1\})$. It means that a closure of $E(L\text{-}\{A_1\})$ on the consequence operation C gives a D-theory to which A_1 belongs again. Thus, it is necessary to replace $E(L-\{A_1\})$ by its relatively minimal decreasing. In view of $A_2,...,A_n$, $E(L-\{A_1,...,A_n\})$ should be replaced by its some relatively minimal subtheory. It means that the set T^+ is reduced by sentences $A_1,...,A_n$ in such a way that all sentences $B_1,...,B_k$ remain still our beliefs. Moreover, both contractions preserve as many sentences from T^+ as possible. Of course, $(T^+\Theta\{A_1,...,A_n\})^1_{B1,...,Bk}$ as well as $(T^+\Theta\{A_1,...,A_n\})^2_{B1,...,Bk}$ are D-theories, because as it was already showed, every relatively minimal R-theory is a D-theory.

The main difference between both contractions is the following: if the set $A_1,...,A_n$ follows from the set $W = \{B_1,...,B_k, B_{k+1},...,B_m\}$ and the set $\{A_1,...,A_n\}$ does not follow from any proper subset of W, then $\{B_{k+1},...,B_m\}\cap nE^{T+}_{B1,...,Bk}(L-\{A_1,...,A_n\}) = \varnothing$ while the set $E^{T+}_{B1,...,Bk}(L-\{A_1,...,A_n\})$ can contain even m-(k+1) sentences from the set $\{B_{k+1},...,B_m\}$. Then, the set of beliefs $(T^+\Theta\{A_1,...,A_n\})^1_{B1,...,Bk}$ is in some sense unknown: it is impossible to know which sentences from the set $\{B_{k+1},...,B_m\}$ remain our beliefs. We only surely know that $\{B_{k+1},...,B_m\} \not\subset (T^+\Theta\{A_1,...,A_n\})^1_{B1,...,Bk}$.

The next difference between both contractions is that, if $\{A_1,...,A_n\}$ independently follows from the set $\{B_1,...,B_m\}$ and from $\{C_1,...,C_s\}$, then $\{C_1,...,C_s\} \cap nE^{T+}_{B1,...,Bk}(L-\{A_1,...,A_n\}) = \varnothing$ while $E^{T+}_{B1,...,Bk}(L-\{A_1,...,A_n\})$ can contain even s-1 elements from the set $\{C_1,...,C_s\}$.

Let us notice that in both cases if $\{A_1,...,A_n\}\cap T^+ = \varnothing$, then T^+ remains a set of our beliefs. Moreover, a removing of any tautology from T converts T into the empty set. These properties are analogous to those for the expansion of truth. If we will add to T any sentence from T, then T will not change. If we will add any counter-tautology to T then T will be converted into the set of all sentences L.

Analogously to the case of expansion, also here one can consider a *contraction of falsehood*. Indeed, if T- is a set of our negative beliefs, and because of some reason sentences $A_1,...,A_n$ become for us already true, then our new set of negative beliefs is either:

$$(T^- \Theta \{A_1,...,A_n\})^1_{B1,...,Bk} = T^- \cap E^{d\,T-}_{B1,...,Bk}(L-\{A_1,...,A_n\})$$

or:

$$(T^- \Theta \{A_1,...,A_n\})^2_{B1,...,Bk} = T^- \cap nE^{d\,T-}_{B1,...,Bk}(L-\{A_1,...,A_n\})$$

where $E^{d\,T}{}_{-B1,...,Bk}(L\text{-}\{A_1,...,A_n\})$ is a R^d-theory for T- minimal relatively to the set $\{B_1,...,B_k\}$.

Usually together with expansion and contraction the third kind of reasoning is consider (cf. [1], [2], [3]), a *revision*, i.e. a procedure adding to the set of beliefs a sentence inconsistent with some sentences from this set. In our approach the revision is an example of two successive steps of reasoning. Thus, the first step of the revision of truth (falsehood) is a contraction of truth (falsehood), and the second step is an expansion of truth (falsehood):

$$(T^+ \otimes \{A\})^i{}_{B1,...,Bk} = (T \ominus \{\neg A\})^i{}_{B1,...,Bk} \oplus \{A\}$$

for $i\{0,1\}$. In the case of truth $T = T^+$, and in the case of falsehood $T = T^-$.

The complete form of the logic enables a formalisation of the non-monotonic reasoning consisting of two monotonic procedures: deductive and reductive; so for the logic of truth as for the logic of falsehood.

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