A-realcompact spaces.

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Abstract

Relations between homomorphisms on a real function algebra and different properties (such as being inverse-closed and closed under bounded inversion) are studied.

1 Introduction and notation

By a function algebra $A$ on $X$ we mean a family of real-valued functions on $X$ such that: 1) $A$ is a linear algebra with unit under operations defined pointwise, 2) $A$ separates points on $X$ and 3) $A$ is closed under bounded inversion, that is, if $f \in A$ and $f \geq 1$, then $\frac{1}{f} \in A$. We denote by $Hom(A)$ the family of all $A$-homomorphisms, that is, non null multiplicative real linear functionals on $A$, endowed with the Gelfand topology.

$Hom(A)$ has been intensively studied when $X$ is a completely regular Hausdorff space and $A$ is $C(X)$ (see [12]). In recent years different papers have been devoted to study homomorphisms on some subalgebras of $C(X)$, for example algebras of differentiable functions have been considered in [1]-[5], [14] and [15]. As can be seen in the quoted papers, in studying function algebras frequently one needs results asserting that a homomorphism is the evaluation at some point of the supporting space. This paper is devoted to elaborate a general theory related with this subject.

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2 Single-set evaluating algebras and A-realcompactness

2.1.- Let \( X \) be a completely regular Hausdorff space, \( Y \subseteq X \) and \( f : Y \to \mathbb{R} \) a continuous map. If \( f \) has a continuous extension to \( p \in X \setminus Y \), this extension will be denoted by \( \hat{f}(p) \). For \( f : X \to \mathbb{R} \), \( Z(f) = \{ x \in X : f(x) = 0 \} \). A set \( S \subseteq Y \) is a zero set if there exists \( g \in C(Y) \) such that \( S = Z(g) \) and \( \overline{S}^X \) is the closure of \( S \) in \( X \). As usual \( \beta X \) denotes the Stone-Čech compactification of \( X \).

2.2.- The elements of any function algebra can be considered as uniformly continuous functions on \( X \) in the following sense. Denote by \( A_b \) the subalgebra of all bounded functions in \( A \). Let \( U_A \) be the uniformity generated on \( X \) by \( A_b \), that is \( U_A \) is defined by the pseudometrics

\[
d_f(x, y) = |f(x) - f(y)|; \quad f \in A_b, x, y \in X.
\]

Let \( \tau_A \) denote the topology induced by \( U_A \) on \( X \). Since \( A \) separates points in \( X \), \( (X, \tau_A) \) is a completely regular Hausdorff space. All topological notions on \( X \) are assumed in the \( \tau_A \) topology.

Denote by \( X_A \) the completion of the uniform space \( (X, U_A) \), then \( X_A \) is a compact Hausdorff space and \( X \) can be considered as a dense subspace of \( X_A \). It is known that each \( f \in A_b \) has a unique continuous extension \( \hat{f} \) to \( X_A \). Set \( \hat{A} = \{ f : f \in A_b \} \). \( \hat{A} \) separates points in \( X_A \) ([7]) then, by the Stone-Weierstrass theorem, \( \hat{A} \) is a dense subspace of \( C(X_A) \) in the uniform norm.

2.3.- The following result from [7] will be used in the sequel:

Theorem. Let \( A \) be a function algebra on \( X \), then

(\( a \)) \( \varphi \in \text{Hom}(A_b) \) if and only if there exists a (unique) \( p \in X_A \) such that \( \varphi(f) = \hat{f}(p) \) for every \( f \in A \). Moreover \( X_A \) is (homeomorphic to) the maximal ideal space of \( A_b \);

(\( b \)) \( \varphi \in \text{Hom}(A) \) if and only if there exists a (unique) point \( p \in X_A \) such that, every \( f \in A \) has a finite continuous extension \( \hat{f}(p) \) to \( p \) and \( \varphi(f) = \hat{f}(p) \). The set \( I(A) \) of all such \( p \), with the topology induced by \( X_A \), is (homeomorphic to) the maximal ideal space of \( A \).
2.4.- In what follows we associate to a given function algebra $A$ the spaces $X_A$ and $I(A)$ defined above. Moreover, we identify $Hom(A)$ with $I(A)$ and $X$ with a (dense) subset of $X_A$. Thus we have the inclusions,

$$X \subset I(A) \subset X_A.$$ 

In studying properties of homomorphisms it is important to have conditions to recognize points in $I(A) \setminus X$. It is easy to verify that for a point $p \in X_A \setminus X$ the following assertions are equivalents:

(a) $p \in I(A)$;

(b) for every $f \in A$, there exists a net $\{x_\lambda\}$ in $X$ such that $x_\lambda \to p$ and $f(x_\lambda)$ is bounded;

(c) for every $f \in A$, there exists a neighbourhood $V$ of $p$ in $X_A$ such that $f(V \cap X)$ is bounded.

2.5.- We need some definitions: a function algebra $A$ on $X$ is called single-set evaluating if, for every $\varphi \in A$ and each $f \in A$, there exists $x \in X$ such that $\varphi(f) = f(x)$. $A$ is called inverse-closed if for every $f \in A$ such that $Z(f) = \emptyset$, $\frac{1}{f} \in A$. It is easy to prove that inverse-closed algebras are single-set evaluating. There exist single-set evaluating algebras which are not inverse-closed [6].

2.6.- Given a nonempty set $X$, $(A,B)$ is called a subordinated pair [7] on $X$ if: i) $A$ and $B$ are function algebras on $X$; ii) $B \subseteq A$; iii) every homomorphism on $B$ has an extension to a homomorphism on $A$.

2.7.- Theorem. For a function algebra $A$ on $X$ the following conditions are equivalent:

(a) $A$ is single-set evaluating;

(b) For all $p \in I(A) \setminus X$, if $f \in A$ and $0 < f \leq 1$, then $f(p) \neq 0$;

(c) $(RA,A)$ is a subordinated pair, where $RA$ the smallest inverse-closed algebra on $X$ containing $A$. 
Proof.

i) Suppose that (a) holds but (b) does not. Fix \( p \in J(A) \setminus X \) and \( h \in A \) such that \( 0 < h \leq 1 \) and \( h(p) = 0 \). Since evaluation at \( p \) is a homomorphism on \( A \), \( A \) is not single-set evaluating.

ii) Suppose that (b) holds and \( A \) is not single-set evaluating. Take \( \varphi \in Hom(A) \), \( p \in f(A) \) and \( k \in A \) such that \( \varphi(g) = \hat{g}(p) \) for every \( g \in A \) and \( \varphi(k) \neq k(x) \) for all \( x \in X \). Set \( h(x) = (k(x) - \varphi(k))^2 \) and \( f(x) = \frac{h(x)}{1 + h(x)} \). Then \( \hat{f}(p) = \varphi(f) = 0 \) and \( 0 < f(x) \leq 1 \). This contradicts (b).

iii) For (a) implies (c) see lemma 16 of [6].

iv) Since \( RA \) is inverse-closed it is single-set evaluating. If \( (RA, A) \) is a subordinated pair, then \( A \) is single-set evaluating.

2.8.- Recall that a completely regular Hausdorff space \( Y \) is realcompact [12] if every \( C(Y) \)-homomorphism is the evaluation at some point \( p \) in \( Y \). This concept can be generalized in the following way: if \( A \) is a function algebra on \( X \), \( X \) is said to be \( A \)-realcompact if every \( A \)-homomorphism is the evaluation at some point \( p \) of \( X \). A similar notion was used in [8], [16] and [17].

2.9.- Remarks.

1) If \( A_b = A \), then \( X \) is \( A \)-realcompact if and only if \( X \) is compact (in the \( \tau_A \) topology). When \( X_A \setminus X \neq \emptyset \) we can obtain \( A \)-realcompactness only when \( A \) contains an unbounded function. In particular if \( (X, \tau) \) is a pseudocompact noncompact, completely regular Hausdorff space and \( A = C(X) \), then \( X \) is not \( A \)-realcompact.

2) Notice that if \( A \) and \( B \) are function algebras on \( X \), \( B \subset A \), with \( X \) \( A \)-realcompact, then \( X \) is \( B \)-realcompact if and only if \( (A, B) \) is a subordinated pair.

2.10.- Proposition. Let \( A \) and \( B \) be function algebras on \( X \) with \( B \) uniformly dense in \( A \). Then \( (A, B) \) is a subordinated pair.
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**Proof.** Since $B_b$ is uniformly dense in $A_b$, the spaces $C(X_A)$ and $C(X_B)$ are isomorphic, thus by the Banach-Stone theorem (see [12]) $X_A$ and $X_B$ are homeomorphic. We may identify $X_A$ and $X_B$. Fix a homomorphism $\varphi$ on $B$ and a point $p \in X_A$ such that for every $f \in B$, $\varphi(f) = \tilde{f}(p)$.

We will finish our proof by showing that every $g \in A$ has a (unique) continuous finite extension to $p$. Fix $g \in A$ and $f \in B$ such that $\sup_{x \in X_A} |f(x) - g(x)| \leq 1$. There exist a neighbourhood $V$ of $p$ in $X_A$ and a positive constant $M$ such that for every $y \in V \cap X$, $|f(y)| \leq M$. Then for every $y \in V \cap X$, $|g(y)| \leq M + 1$, now the assertion follows from 2.4.

In [10] (proposition 1.8) was proved the following fact: if $X$ is a realcompact space and $A \subset C(X)$ is a subalgebra with unit, closed under bounded inversion, uniformly dense in $C(X)$, then $\text{Hom}(A) = X$.

Our next result, as an application of proposition 2.10 (see remark 2.9.2), provides a natural extension.

2.11.- Corollary. Let $A$ and $B$ be function algebras on $X$, $B \subset A$. If $B$ is uniformly dense in $A$ and $X$ is $A$-realcompact, then $X$ is $B$-realcompact.

2.12.- Theorem. Let $A$ be a single-set evaluating algebra on $X$. Then $X$ is $A$-realcompact if and only if $X$ is $\text{RA}$-realcompact (see (e) in 2.7). Moreover if $A$ is inverse-closed, then $X$ is $A$-realcompact if and only if for every $p \in X_A \setminus X$, there exists

$$f \in A_b, \quad 0 < f \leq 1, \quad \text{such that } \tilde{f}(p) = 0. \quad (1)$$

**Proof.** The first part follows from theorem 2.7, the remark 2) in 2.9 and the construction of $\text{RA}$.

For the second part suppose first that $X$ is $A$-realcompact. Suppose that $p \in X_A \setminus X$. Taking into account that $p \notin I(A) = X$, there exists $f \in A \setminus A_b$ such that for every net $\{x_\lambda\}$ in $X$, with $x_\lambda \to p$, $f(x_\lambda)$ is unbounded (see the last assertion in 2.4). Then $h(p) = 0$ and $0 < h(x) \leq 1$ for $x \in X$, where $h(x) = \frac{1 + f(x)}{1 + f^2(x)}$.

Suppose now that for all $p \in X_A \setminus X$ there exists $f \in A$ such that $0 < f \leq 1$ and $\tilde{f}(p) = 0$. By defining $g(x) = \frac{1}{f(x)}$, we have that $g \in A$.
and for every net \( \{x_\lambda\} \) in \( X \), \( x_\lambda \to p \), \( \{g(x_\lambda)\} \) is not bounded. This completes the proof.

2.13.- Remark. In general condition (1) does not imply \( A \)-realcompactness. For example, let \( X \) be the real interval \((0,1] \) and \( A \) the restriction of continuous functions in \([0,1] \) to \((0,1] \). In this case the condition holds but \( X \) is not \( A \)-realcompact (notice that \( X_A = [0,1] \)).

2.14.- Theorem. Let \( A \) be a function algebra. Then \( X_A \) is the Stone-Čech compactification of \( X \) if and only if for any disjoint zero sets \( S \) and \( T \) in \( X \), there exists \( f \in A \), such that

\[
0 \leq f \leq 1, \quad f(S) = \{0\} \quad \text{and} \quad f(T) = \{1\}. \tag{2}
\]

Proof. If \( A \) satisfies (2) by theorem 11 of [11], \( A_b \) is uniformly dense in the space \( C_b(X) \) of all real continuous bounded functions on \( X \), then \( \beta X = X_A \).

On the other hand if \( \beta X = X_A \), \( A_b \) is dense in \( C_b(X) \) and the result follows again from theorem 11 of [11].

From theorems 2.12 and 2.14 we obtain a proof of the following result due to S. Mrówka (proposition 3.11.10 in [9]).

2.15.- Corollary. Let \( X \) be a completely regular Hausdorff space. Then \( X \) is realcompact if and only if for every \( p \in \beta X \setminus X \), there exists \( f \in C(X) \) such that \( 0 < f(x) \leq 1 \), \( x \in X \), and \( f(p) = 0 \).

The next result extends Theorem 2 of [15]. Jaramillo presented in [15] different examples of function algebras for which Theorem 2.16 may be applied.

2.16.- Theorem. Let us suppose that a function algebra \( A \) on \( X \) satisfies the following conditions:

(a) for every \( f, g \in A \) and \( \rho, \epsilon > 0 \), if the sets

\[
P_\epsilon(f) = \{x : |f(x)| \leq \epsilon\} \quad \text{and} \quad Q_\rho(g) = \{x : |g(x)| \geq \rho\}
\]


are not empty and disjoint, there exists \( h \in A \), \( 0 \leq h \leq 1 \), such that
\[
h(P_\varepsilon(f)) = \{0\} \text{ and } h(Q_\rho(g)) = \{1\};
\]

(b) given an open (in the \( \tau_A \) topology) cover \( \{H_n\} \) of \( X \), such that
\[
\overline{H_n} \subseteq H_{n+1}, \text{ and } f : X \to \mathbb{R}, \text{ if there exists a sequence } \{f_n\} \text{ in } A
\]
such that \( f_n \mid_{H_n} = f \mid_{H_n}, \text{ then } f \in A; \)

(c) for every \( p \in X_A \setminus X \), there exists \( g \in C(X_A) \) which satisfies (1).

Then \( X \) is \( A \)-realcompact.

**Proof.** Let \( \varphi \) be a homomorphism on \( A \). There exists \( p \in X_A \) such that
\[
\varphi(f) = \hat{f}(p) \text{ for every } f \in A. \text{ We will show that } p \in X.
\]
Suppose that \( p \in X_A \setminus X \), take \( g \in C(X_A) \) such that \( 0 < g \leq 1 \) and \( \hat{g}(p) = 0 \). Set
\[
E_n = \{ x \in X_A : g(x) > \frac{1}{2^n} \}, \ n = 1, 2, ...
\]

We may suppose that each \( E_n \) is not empty. Since \( \hat{A} \) is dense in \( C(X_A) \), there exists a sequence \( \{f_n\} \) in \( A_0 \) such that
\[
\| \hat{f}_n - g \|_\infty \leq \frac{1}{2^{n+3}} \text{ and } \| \hat{f}_n - \hat{f}_{n+1} \|_\infty \leq \frac{1}{2^{n+3}},
\]
where \( \| . \|_\infty \) denotes the sup norm in \( C(X_A) \). Set
\[
F_n = \{ x \in X_A : \hat{f}_n(x) \geq \frac{1}{2^n} \}.
\]
It is easy to prove that for \( n \geq 2, E_{n-1} \subseteq F_n \subseteq E_{n+1}. \)

Now we have that \( (X \cap \bigcup_{n \in \mathbb{N}} E_n) = \bigcap_{n \in \mathbb{N}} (X \setminus F_n) \) thus \( \{F_{2n} \cap X\} \) is an increasing open cover of \( X \). For each \( n \geq 2 \) take \( g_n \in A, 0 \leq g_n \leq 1 \), such that
\[
g_n(F_{2n+2} \cap X) = \{1\} \text{ and } g_n(F_{2n} \cap X) = \{0\}.
\]
Notice that \( \hat{g}_n(p) = 1 \), thus \( \varphi(\hat{g}_n) = 1 \). The function \( f(x) = \sum_{n=2}^{\infty} g_n(x), \)
\( x \in X \) is well defined. Set \( k_n(x) = \sum_{j=2}^{n} g_j(x) \). Since \( k_n \in A, f \in A. \)
It is easy to see that for every \( x \in X \) and each \( n \), \( k_n(x) \leq f(x) \), then 
\[
\varphi(f) \geq \varphi(k_n) = \sum_{j=1}^{n} \varphi(g_j) = n \quad (\text{see 1.4 of [13]}),
\]
this says that \( \varphi(f) = \infty \), a contradiction.

2.17.- Theorem 2.3 gives a representation of the real maximal ideal of \( A \) but, as the following result will prove, we can not expect to obtain a one to one relation between \( z \)-ultrafilters and maximal ideals. The notion on \( z \)-filter is used as in [12]. An ideal in \( A \) is a proper ideal. For an ideal \( I \), 
\[
Z(I) = \{Z(f) : f \in I\}.
\]
If \( J \) is a \( z \)-filter \( J^{-1}_A = \{f \in A : Z(f) \in J\} \).

2.18.- Theorem. Let \( A \) be a function algebra which satisfies (2). The following assertion are equivalent:

(a) for each maximal ideal \( I \) in \( A \), there exists \( p \in \beta X \) such that 
\[
I = \{f \in A : p \in \overline{Z(f)}^{\beta X}\}.
\]

(b) for each maximal ideal \( I \) in \( A \), there exists a maximal ideal \( J \) in \( C(X) \) such that \( I \subset J \);

(c) for each maximal ideal \( I \) in \( A \), \( Z(I) \) is a \( z \)-ultrafilter;

(d) \( A \) is inverse-closed.

Proof. Since \( A \) satisfies (2), for every zero set \( P \) in \( X \) there exists \( f \in A \) such that \( Z(f) = P \).

The assertions (a) implies (b) and (b) implies (a) follow directly from the Gelfand-Kolmogorov theorem ([12], 7.3).

(b) implies (c) Fix maximal ideals \( I \) and \( J \) in \( A \) and \( C(X) \) respectively, with \( I \subset J \). \( Z^{-1}_A(Z(J)) \) is an ideal in \( A \). Therefore, \( I = Z^{-1}_A(Z(J)) \). Since \( Z(I) = Z(J), Z(I) \) is a \( z \)-ultrafilter.

(c) implies (b) Fix a maximal ideal \( I \) in \( A \), since \( Z(I) \) is a \( z \)-ultrafilter 
\[
J = \{f \in C(X) : Z(f) \in Z(I)\}
\]
is a maximal ideal in \( C(X) \) containing \( I \).

(c) implies (d) Take \( f \in A \) such that \( Z(f) = \emptyset \) and set 
\[
I = \{gf : g \in A\}.
\]
Since \( f \in I \), \( I \) can not be an ideal, therefore \( I = A \).

(d) implies (c) Fix an ideal \( I \) in \( A \). Since \( A \) is inverse closed \( \emptyset \notin Z(I) \).

On the other hand, if \( f, g \in I \) and \( h \in A \), \( Z(f^2 + g^2) = Z(f) \cap Z(g) \) and 
\[
Z(f) \subset Z(fg) = Z(g).
\]
3 The sequentially evaluating property

3.1.- A function algebra $A$ on $X$ is called *sequentially evaluating* if, for every $\varphi \in \text{Hom}(A)$ and each sequence $\{f_n\}$ in $A$, there exists $x \in X$ such that $\varphi(f_n) = f_n(x)$, for $n = 1, 2, \ldots$ This property has been intensively studied in [2]. As far as we know the use of this property goes back to S. Mazur (see the note to statement $A$ of [8]). If a function algebra $A$ on $X$ has the sequentially evaluating property, then every homomorphism on $A$ is sequentially continuous on $A^\circ$, where $A^\circ$ is the algebra $A$ endowed with the pointwise convergence topology. This fact was noticed for some particular algebras in [2] and [6].

3.2.- Denote by $[A \cup C(X)]$ the closed under bounded inversion algebra on $X$ generated by $A$ and $C(X)$. By setting

$$A_1 := \{ \sum_{k=1}^{n} f_k g_k : f_k \in A, g_k \in C(X), n \in \mathbb{N} \},$$

we have that $[A \cup C(X)] = \{ h_1/h_2 : h_1, h_2 \in A_1, h_2 \geq 1 \}$.

3.3.- Theorem. Let $A$ be a single-set evaluating algebra on $X$. The following conditions are equivalent:

(a) $A$ has the sequentially evaluating property.

(b) Each zero set in $X \setminus X$ does not meet $I(A)$.

(c) $[A \cup C(X)]$ is single-set evaluating.

Proof. Suppose that (a) holds and (b) fails, then there exists a zero set $P \subset X \setminus X$ such that $P \cap I(A) \neq \emptyset$. Fix $q \in P \cap I(A)$ and let $\varphi$ be the evaluation at $q$. Since $P$ is a zero set, there exists $f \in C(X)$ such that $P = Z(f)$. Since $A$ is dense in $C(X)$ for the uniform norm, there exists $\{f_n\}$ in $A_b$, with $f_n \to f$ uniformly on $X$. We have that $\varphi(f_n) = f_n(q) \to f(q) = 0$. Set $g_n = f_n - \varphi(f_n) \in A_b$. According to the above arguments $g_n \to f$ uniformly on $X$ and $\varphi(g_n) = 0$. By the sequentially evaluating property there exists $x_0 \in X$ such that $\varphi(g_n) = g_n(x_0) = 0$. This says that $\lim_{n} g_n(x_0) = f(x_0) = 0$ and we have a contradiction.

(b) implies (c) Suppose that (b) holds and let $\varphi$ be a homomorphism on $[A \cup C(X)]$. We will prove that for each $h \in [A \cup C(X)]$, ...
$Z(h - \varphi(h)) \neq \emptyset$. Since $\varphi$ is a homomorphism on $A (C(X_A))$, there exists $p \in I(A) (g \in C(X_A))$ such that, for each $f \in A \ (g \in C(X_A))$ $\varphi(f) = \hat{f}(p) (\varphi(g) = \hat{g}(q))$. Since $A_b \subset A \cap C(X_A)$, for each $f \in A_b$, $\hat{f}(p) = \hat{f}(q)$. Taking into account that $\hat{A}$ separates points in $X_A$, we have that $p = q$. Now if $f \in (A \cup C(X_A))$, set $g_f = f - \varphi(f)$. If $Z(g) \cap X = \emptyset$, then $Z(g) \cap I(A) = \emptyset$ and this is not possible ($p \in Z(g) \cap I(A)$).

Since for every $f \in A$, $(f - \varphi(f))^2$ has a continuous extension to $X_A$, we have that for any $h \in A_1$ (see 3.2), $Z(h - \varphi(h)) \neq \emptyset$. In fact, if $f_1, ..., f_n \in A$ and $g_1, ..., g_n \in C(X_A)$,

\[
Z(h - \varphi(h)) = \left\{ \sum_{k=1}^{n} \frac{(f_k - \varphi(f_k))^2}{1 + (f_k - \varphi(f_k))^2} + (g_k - \varphi(g_k))^2 \right\}
\]

Using the continuity of $f_k - \varphi(f_k)$ and $g_k - \varphi(g_k)$, we have that for any net $\{x_\lambda \}_{\lambda \in \Lambda}$ in $X$, such that $x_\lambda \to p$ in $X_A$,

\[
\lim_{\lambda} h(x_\lambda) = \sum_{k=1}^{n} \lim_{\lambda} f_k(x_\lambda) \lim_{\lambda} g_k(x_\lambda) = \sum_{k=1}^{n} \hat{f}_k(p) \hat{g}_k(p) = \hat{h}(p).
\]
Finally, if \( h = \frac{h_1}{h_2} \in [A \cup C(X_A)] \) with \( h_1, h_2 \in A_1 \) (\( h_2 \geq 1 \)), set \( \hat{h}(p) = \frac{h_1(p)}{h_2(p)} \). Then, by defining \( \varphi(h) = \hat{h}(p) \) for \( h \in [A \cup C(X_A)] \), we have that \( \varphi \in Hom([A \cup C(X_A)]) \) and \( \varphi(f) = \psi(f) \) for \( f \in A \).

Now, fix a sequence \( \{f_n\} \) in \( A \). Set \( g_n(x) = \frac{1}{2^n} \frac{(f_n(x) - \varphi(f_n))^2}{1 + (f_n(x) - \varphi(f_n))^2} \) and \( g = \sum_{n=1}^{\infty} g_n \). We have that \( \hat{g} \in C(X_A) \). Let us prove that \( \varphi(g) = 0 \).

In fact, notice that the sequence \( \left\{ \sum_{k=1}^{n} g_k \right\} \) converges uniformly to \( g \) and \( \sum_{k=1}^{n} g_k \leq g \). Then, given \( \epsilon > 0 \) and \( n \) such that \( \| \sum_{k=1}^{n} g_k - g \|_{\infty} < \epsilon \), it follows that

\[
0 = \varphi\left( \sum_{k=1}^{n} g_k \right) \leq \varphi(g) = \varphi(g - \sum_{k=1}^{n} g_k) \leq \epsilon \varphi(1) = \epsilon.
\]

Taking into account that \( [A \cup C(X_A)] \) is single-set evaluating, there exist \( x_0 \in X \) such that \( 0 = \varphi(g) = g(x_0) \). Therefore \( \varphi(f_n) = f_n(x_0) \) for each \( n \).

3.4. **Remark.** If \( A \) is an inverse-closed algebra on \( X \) closed under the uniform convergence, then \([A \cup C(X_A)] = A\), and \( A \) has the sequential evaluating property. This assertion can be obtained from the result of S. Mazur quoted in [8] and gives a proof of following fact: \( X \) need not be \( A \)-realcompact when \( A \) is a sequentially evaluating algebra on \( X \). For certain class of algebras the sequentially evaluating property implies \( A \)-realcompactness (for example if \( X \) is a Lindelöf space in the \( \tau_A \) topology), this just was the main reason for studying this property in [2].

The last proposition in this section can be proved as theorem 2.16.

3.5. **Proposition.** If a function algebra \( A \) satisfies conditions (a) and (b) in theorem 2.16 then \( A \) has the sequentially evaluating property.

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