On a reaction-diffusion system involving the critical exponent.

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Abstract

In this paper we study the existence and multiplicity of the nontrivial solutions for the following elliptic system with Dirichlet boundary conditions and critical nonlinearity

\[
\begin{cases}
-\Delta u = \lambda u + W(x)u |u|^2 - 2u - kv & \text{in } \Omega \\
-\Delta v = \delta u - \gamma v & \text{in } \Omega \\
u = v = 0 & \text{on } \partial \Omega
\end{cases}
\]

where \( \Omega \subset \mathbb{R}^N (N \geq 3) \) is a bounded regular domain, \( W(\cdot) \in L^\infty(\Omega) \) with the property that there exists \( \eta > 0 \) such that \( W(\cdot) \geq \eta \) a.e. in \( \Omega \) and \( \lambda, \delta, \gamma \) are real parameters. We show that the number of nontrivial solutions, in a left neighbourhood of each \( \lambda_j, j = 1, 2, \ldots \), is at least twice the multiplicity of \( \lambda_j \), where the set \( \{\lambda_j\}_{j \in \mathbb{N}} \) represents the spectrum of a certain integro-differential operator.

1 Introduction

Rothe in [R] considered the system of reaction diffusion equations

\[
\begin{cases}
\partial_u \partial_t = \mu \Delta u + f(u) - v \\
\varepsilon \partial_v \partial_t = \Delta v + u - v
\end{cases}
\]  

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for \((t, x) \in (0, \infty) \times \Omega\). Here \(u, v\) are real functions of \((t, x) \in [0, \infty) \times \Omega\), where \(\Omega \subset \mathbb{R}^N (N \geq 1)\) is open, bounded and connected. As explained in [RM], \(u\) and \(v\), which are called the activator and inhibitor respectively, can be interpreted as relative concentrations of substances known as morphogens. The system (1) is supplemented by Dirichlet boundary conditions

\[ u = v = 0, \text{ for } (t, x) \in (0, \infty) \times \partial \Omega \]

and the initial conditions

\[ u(0, x) = u_0(x), \quad v(0, x) = v_0(x), \text{ for all } x. \]

As shown in [RM], the existence of equilibrium solutions in (1) is determined by the problem with \(\varepsilon = 0\) and the equilibrium states are solutions of the elliptic system

\[
\begin{aligned}
\mu \Delta u + f(u) - v &= 0 \quad \text{in } \Omega \\
\Delta v + u - v &= 0 \quad \text{in } \Omega
\end{aligned}
\]

subject to Dirichlet boundary conditions

\[ u = v = 0 \text{ on } \partial \Omega. \]

It will be convenient to split the function \(f\), which models autocatalytic and saturation effects, into the linear and higher order terms

\[ f(u) = \lambda u + g(u). \]

**Notation.** In the rest of the paper we make use of the following notation

- \(L^p(\Omega), 1 \leq p \leq \infty\), denote Lebesgue spaces; the norm in \(L^p\) is denoted by \(\|\cdot\|_p\);
- \(W^{k,p} (\Omega)\) denote Sobolev spaces;
- \(H^1_0(\Omega)\) denotes \(W^{1,2}_0 (\Omega)\), endowed with the norm \(\|u\|^2 = \int_{\Omega} |\nabla u|^2 \, dx\);
- \(H^{-1}(\Omega)\) denotes the topological dual of \(H^1_0(\Omega)\); the norm in this space is denoted by \(\|\cdot\|_{H^{-1}}\).

We consider below the problem of finding nontrivial solutions of the slightly more general elliptic system with Dirichlet boundary conditions and critical nonlinearity

\[
\begin{aligned}
-\Delta u &= \lambda u + W(x)u|u|^{2^* - 2} - kv \quad \text{in } \Omega \\
-\Delta v &= \delta u - \gamma v \quad \text{in } \Omega \\
u &= v = 0 \quad \text{on } \partial \Omega
\end{aligned}
\]

\((P)\)
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where $\Omega \subset \mathbb{R}^N (N \geq 3)$ is a bounded regular domain, $\delta$, $\gamma$ and $k$ are constants such that $k \delta > 0$ and $\gamma > - \lambda_1 (\Omega)$, where $\lambda_1 (\Omega)$ is the first eigenvalue of the Dirichlet Laplacian on $\Omega$, and $W(\cdot) \in L^\infty (\Omega)$ with the property that there exists $\eta > 0$ such that $W(\cdot) \geq \eta$ a.e. in $\Omega$. Here $2^* = 2N / (N - 2)$.

In the subcritical case the system (1) has been studied by various authors (see [Ro], [Si], [FM], [NT] and others). The review, even partial, of their results is out of the scope of this paper.

Assuming $u$ to be known, the Dirichlet boundary value problem

$$
\begin{align*}
- \Delta v + \gamma v &= \delta u \quad \text{in } \Omega \\
v &= 0 \quad \text{on } \partial \Omega
\end{align*}
$$

is uniquely solved by $v = 1k Bu$ where the operator $B = k \delta (-\Delta + \gamma)^{-1}$ is bounded from $L^p (\Omega)$ to $W^{2,p}(\Omega)$ for all $1 \leq p < \infty$. Also, by the Schauder theory, $B$ maps the Hölder space $C^\alpha (\Omega)$ into $C^{1+\alpha} (\Omega)$.

Moreover, it is easily checked that $B$ is positive and self-adjoint in the sense that

$$
\int_\Omega uBu dx = \frac{1}{k \delta} \int_\Omega |\nabla w|^2 + \gamma w^2 dx
$$

for $u \in L^2 (\Omega)$ and $w = Bu$; and if $w = Bu$, $z = Bv$ then

$$
\int_\Omega uBu dx = \frac{1}{k \delta} \int_\Omega \nabla w \nabla z + \gamma wz dx = \int_\Omega uBu dz.
$$

Let us define the operator

$$
T \equiv - \Delta + B : L^2 (\Omega) \to L^2 (\Omega), \text{ with } D(T) = W^{2,2} (\Omega) \cap H^1_0 (\Omega).
$$

It is easy to observe that $T$ is symmetric on its domain $D(T)$ i.e.

$$
\langle Tu_1, u_2 \rangle = \langle u_1, Tu_2 \rangle \text{ for all } u_1, u_2 \in D(T),
$$

where $\langle \cdot, \cdot \rangle$ denotes the $L^2$-inner product.

If $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots$ and $(\varphi_k)_k$ denote respectively the eigenvalues and the eigenfunctions of $-\Delta$ in $\Omega$ under zero Dirichlet boundary conditions, then one can verify easily that the $\varphi_k$'s are also eigenfunctions of $T$ corresponding to the modified eigenvalues

$$
\tilde{\lambda}_k = \lambda_k + \frac{k \delta}{\gamma + \lambda_k}, \quad k = 1, 2, \ldots.
$$
A more detailed analysis shows that the spectrum $\sigma(T)$ of $T$ consists precisely of these eigenvalues (see [FM, Corollary 1.2]).

From the above, we obtain that (P) is equivalent to the integro-differential equation

\[ \begin{cases} -\Delta u + Bu = \lambda u + W(x)u|u|^{2^*-2} & \text{in } \Omega \\ u = 0 & \text{on } \partial \Omega \end{cases} \]

We associate to the problem (P') the functional

\[ I_\lambda(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 + uBu - \lambda u^2 dx - \frac{1}{2^*} \int_\Omega W(z)|u|^{2^*} dx, \forall u \in H^1_0(\Omega). \]

In a standard way we can prove that $I_\lambda \in C^1(H^1_0(\Omega),\mathbb{R})$ and the critical points of $I_\lambda$ are solutions of (P').

Note that $p = 2^*$ is the limiting Sobolev exponent for the embedding $H^1_0(\Omega) \hookrightarrow L^p(\Omega)$. Since this embedding is not compact, the functional $I_\lambda$ does not satisfy the Palais-Smale condition in the energy range $(-\infty, +\infty)$. Hence there are serious difficulties when trying to find critical points by standard variational methods.

Using the ideas of Pohozaev (see [P]), Figueiredo and Mitidieri obtained a similar identity for the system (P) (see [FM, Lemma 4.1 and Remark 2.7]). From this identity, if $\Omega$ is starshaped, we can obtain that (P) admits only the trivial solution $u \equiv v \equiv 0$ for $\lambda \leq 0$.

Denote

\[ S_B = \inf_{u \in H^1_0(\Omega) \setminus \{0\}} \frac{\|u\|_{L^2}^2}{\|u\|_{H^1}^2}, \]

where $\|u\|_{L^2}^2 = \int_\Omega |\nabla u|^2 + uBu dx, \forall u \in H^1_0(\Omega)$. From the positivity of $B$ we have that

\[ S_B \geq S = \inf_{u \in H^1_0(\Omega) \setminus \{0\}} \frac{\|u\|^2}{\|u\|_{H^1}^{2^*}}, \]

where $S$ corresponds to the best constant for the Sobolev continuous embedding $H^1_0(\Omega) \hookrightarrow L^{2^*}(\Omega)$. Then $S_B > 0$ because it is well known that $S > 0$.

Under the above conditions and notations, the result proved in this paper is the following:
Theorem 1.1. For $\lambda > 0$ denote $\overline{\lambda}_+ = \min \{ \lambda_j : \lambda < \lambda_j \}$ and suppose that the multiplicity of $\overline{\lambda}_+$ is $m$. Then, if

$$\overline{\lambda}_+ - \lambda < \left( \frac{\eta}{\|W\|_\infty} \right)^{\frac{2}{N}} S_B \left( \text{meas} (\Omega) \right)^{-2/N},$$

the problem $(P)$ admits at least $m$ pairs of nontrivial solutions

$$\{(u_k(\lambda), v_k(\lambda)); (-u_k(\lambda), -v_k(\lambda))\}, \ k = 1, 2, \ldots, m.$$ Moreover

$$\|u_k(\lambda)\| \to 0 \text{ and } \|v_k(\lambda)\| \to 0, \text{ as } \lambda \nearrow \overline{\lambda}_+,$$

for every $k \in \{1, 2, \ldots, m\}$.

The proof of the above theorem uses standard ideas and the techniques are essentially the same as those used in [CFS] and [CFP]. The main tool used is the following slightly modified result of Bartolo, Benci and Fortunato (see [BBF, Theorem 2.4]) contained in [CFS, Theorem 2.5]:

Theorem 2.2. Let $H$ be a real Hilbert space with norm $\| \cdot \|_H$ and suppose $I \in C^1 (H, \mathbb{R})$ is a functional on $H$ satisfying the following conditions:

I1) $I$ is even, $I (0) = 0$;

I2) There exists a constant $\beta > 0$ such that the Palais-Smale condition (PS) holds in $(0, \beta)$;

I3) There exist two closed subspaces $V, W \subset H$ and positive constants $\rho, \xi, \beta'$ with $\xi < \beta' < \beta$ such that

i) $I (u) \leq \beta'$ for any $u \in W$;

ii) $I (u) \geq \xi$ for any $u \in V, \|u\|_H^2 = \rho$;

iii) $\text{codim} V < \infty$ and $\text{dim} W \geq \text{codim} V$.

Then there exists at least $\text{dim} W - \text{codim} V$ pairs of critical points of $I$ with critical values belonging to the interval $[\xi, \beta']$. 
2 Proof of Theorem 1

Step 1.
First we show that although the Palais-Smale condition does not hold globally for \( I_\lambda \) it is satisfied locally in \( (-\infty, 1 NS_B^{N/2}||W||_{\infty}^{N/2}) \) in the following sense:

If \( c < 1 NS_B^{N/2}||W||_{\infty}^{N/2} \) and \( (u_m)_{m \geq 1} \) is a sequence in \( H^1_0(\Omega) \) such that

\[
\begin{align*}
I_\lambda (u_m) &\to c \\
\text{d}I_\lambda (u_m) &\to 0 \text{ strongly in } H^{-1}(\Omega),
\end{align*}
\]

then \( (u_m)_{m \geq 1} \) contains a subsequence converging strongly in \( H^1_0(\Omega) \).

Let \( c \in \left(-\infty, 1 NS_B^{N/2}||W||_{\infty}^{N/2}\right) \) and let \( (u_m)_{m \geq 1} \subset H^1_0(\Omega) \) be a sequence such that

\[
I_\lambda (u_m) \to c, \text{ as } m \to \infty, \text{ and}
\]

\[
\text{d}I_\lambda (u_m) \to 0, \text{ as } m \to \infty, \text{ in } H^{-1}(\Omega).
\]

It is easy to observe that there exists \( M > 0 \) a positive constant such that, for every \( m \in \mathbb{N}^*, |I_\lambda (u_m)| \leq M \).

If we choose \( \theta \in (12, 12) \) and \( m \in \mathbb{N}^* \) sufficiently large, we obtain

\[
M + \theta \|u_m\| \geq I_\lambda (u_m) - \theta \text{d}I_\lambda (u_m) u_m \geq \frac{1}{2} \int_{\Omega} |\nabla u_m|^2 + u_m B u_m - \lambda u_m^2 dx-
\]

\[
- \frac{1}{2^*} \int_{\Omega} W(x) |u_m|^{2^*} dx - \theta \int_{\Omega} |\nabla u_m|^2 + u_m B u_m - \lambda u_m^2 dx + \theta \int_{\Omega} W(x) |u_m|^{2^*} dx \geq
\]

\[
\geq \left( \frac{1}{2} - \theta \right) \int_{\Omega} |\nabla u_m|^2 + u_m B u_m - \lambda u_m^2 dx + \left( \theta - \frac{1}{2^*} \right) \int_{\Omega} W(x) |u_m|^{2^*} dx \geq
\]

\[
\geq \left( \frac{1}{2} - \theta \right) \|u_m\|^2 - C_1 \lambda \|u_m\|_{2^*}^2 + \eta \left( \theta - \frac{1}{2^*} \right) \|u_m\|_{2^*}^2 \geq
\]

\[
\geq \left( \frac{1}{2} - \theta \right) \|u_m\|^2 + \inf_{\rho \geq 0} \left[ \eta \left( \theta - \frac{1}{2^*} \right) \rho^{2^*} - C_1 \lambda \rho^2 \right],
\]

where \( C_1 > 0 \) is a positive constant.
Then \((u_m)_{m \geq 1}\) is bounded in \(H^1_0(\Omega)\). Hence we may extract a sub-sequence \((u_m)_{m \geq 1}\) (re-labeled) such that

\[
\begin{align*}
&u_m \rightharpoonup u \text{ weakly in } H^1_0(\Omega) \\
&u_m \rightarrow u \text{ strongly in } L^p(\Omega), \text{ for any } p \in [1, 2^*) \\
&u_m \rightarrow u \text{ a.e. in } \Omega
\end{align*}
\]

Now, we prove that \(u\) is a solution of \((P')\). Let \(\varphi \in C_0^\infty(\Omega)\). Then

\[|dI_\lambda(u)\varphi| \leq ||dI_\lambda(u_m)||_{H^{-1}} ||\varphi|| + |(dI_\lambda(u) - dI_\lambda(u_m))\varphi| \rightarrow 0, \text{ as } m \rightarrow \infty.\]

Hence \(u\) weakly solves \((P')\).

Let \(v_m = u_m - u\). Clearly

\[
\begin{align*}
&v_m \rightharpoonup 0 \text{ weakly in } H^1_0(\Omega) \quad (2) \\
&v_m \rightarrow 0 \text{ strongly in } L^p(\Omega), \text{ for any } p \in [1, 2^*) \quad (3) \\
&v_m \rightarrow 0 \text{ a.e. in } \Omega
\end{align*}
\]

From (2) and (3) observe that

\[
o(1) = dI_\lambda(u_m) v_m = \int_{\Omega} \nabla u_m \nabla v_m + v_m B u_m - \lambda u_m v_m \text{d}x - \int_{\Omega} W(x) v_m u_m |u_m|^{2^* - 2} \text{d}x \\
- \int_{\Omega} |\nabla v_m|^2 + v_m B v_m \text{d}x - \int_{\Omega} W(x) v_m u_m |u_m|^{2^* - 2} \text{d}x + o(1) \\
= ||v_m||_B^2 - \int_{\Omega} W(x) v_m u_m |u_m|^{2^* - 2} \text{d}x + o(1).
\]

Hence

\[
||v_m||_B^2 = \int_{\Omega} W(x) v_m u_m |u_m|^{2^* - 2} \text{d}x + o(1) \leq ||W||_\infty \int_{\Omega} |v_m|^{2^*} \text{d}x + o(1). \quad (4)
\]

Since

\[dI_\lambda(u_m) u_m = o(1),\]
we have that
\[
\int_{\Omega} W(x) |u_m|^2 \, dx = \int_{\Omega} |\nabla u_m|^2 + u_m B u_m - \lambda u_m^2 \, dx + o(1).
\]

Using this last equality we obtain
\[
I_\lambda (u_m) = \frac{1}{2} \left( \|u_m\|_B^2 - \lambda \|u_m\|_2^2 \right) - \frac{1}{2} \int_{\Omega} W(x) |u_m|^2 \, dx \geq \frac{\eta}{N} \|u\|_{2^*}^2 + \frac{1}{N} \|v_m\|_B^2 + o(1) \geq \frac{1}{N} \|v_m\|_B^2 + o(1).
\]

Then
\[
\|v_m\|_B^2 \leq NI_\lambda (u_m) + o(1) < S_B^{N/2} \|W\|_\infty^{N+2} , \text{ for } m \text{ sufficiently large. (5)}
\]

From (4) we have
\[
\|v_m\|_B^2 \leq \|W\|_\infty S_B^{2^*} \|v_m\|_B^{2^*} + o(1) \iff \|v_m\|_B^2 \left( S_B^{2^*} - \|W\|_\infty \|v_m\|_B^{2^* - 2} \right) \leq o(1).
\]

Since, from (5),
\[
S_B^{2^*} > \|W\|_\infty \|v_m\|_B^{2^* - 2} \text{ for } m \text{ large enough,}
\]

we obtain that
\[
v_m \to 0, \text{ strongly in } H_0^1 (\Omega) , \text{ as } m \to \infty,
\]

and this ends the proof of the fact that $I_\lambda$ satisfies the Palais-Smale condition on $(-\infty, 1N S_B^{N/2} \|W\|_\infty^{N+2})$.

**Step 2.**

Set
\[
H_1 = \overline{\lambda_j \geq \lambda_{\pm} + M (\lambda_j)} \text{ and } H_2 = \overline{\lambda_j \leq \lambda_{\pm} + M (\lambda_j)},
\]
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where $M\left(\lambda_j\right)$ denotes the eigenspace of $T$ corresponding to the eigenvalue $\lambda_j$. Denote $\beta_\lambda = H_2 \sup I_\lambda$ and observe that, if $u = \sum_{\lambda_i \leq \lambda_+} a_i \varphi_i \in H_2$, we have

$$I_\lambda(u) = \frac{1}{2} \|u\|^2_B - \lambda \|u\|^2 - \frac{1}{2} \int_\Omega W(x) |u|^2 \, dx \leq \frac{1}{2} \left(\lambda_+ - \lambda\right)$$

$$\int_\Omega u^2 dx - \frac{\eta}{2\varepsilon} \|u\|_{\varepsilon}^2 \leq \frac{1}{2} \left(\lambda_+ - \lambda\right) \left(\text{meas}(\Omega)\right)^{2/N} \|u\|^2_{\varepsilon} - \frac{\eta}{2\varepsilon} \|u\|_{\varepsilon}^2.$$ 

$$\leq \rho \geq 0 \sup \left[\frac{1}{2} \left(\lambda_+ - \lambda\right) \left(\text{meas}(\Omega)\right)^{2/N} \rho^2 - \frac{\eta}{2\varepsilon} \rho^2\right]$$

$$= \frac{1}{N} \eta^{2/N} \left(\lambda_+ - \lambda\right)^{N/2} \left(\text{meas}(\Omega)\right).$$

Thus

$$\beta_\lambda \leq \frac{1}{N} \eta^{2/N} \left(\lambda_+ - \lambda\right)^{N/2} \left(\text{meas}(\Omega)\right).$$

If $u = \sum_{\lambda_i \geq \lambda_+} a_i \varphi_i \in H_1$, a simple computation shows that

$$I_\lambda(u) \geq \left(1 - \frac{\lambda}{\lambda_+}\right) \|u\|^2_B - C_2 \|u\|^2_{\varepsilon},$$

where $C_2 > 0$ is a positive constant. Clearly, there exist constants $\rho_\lambda, \xi_\lambda \in (0, \beta_\lambda)$ such that

$$I_\lambda(u) \geq \xi_\lambda, \text{ for any } u \in H_1, \|u\|_B = \rho_\lambda.$$

**Step 3.**

Now, it is easy to observe that the hypothesis of Theorem 2 are satisfied for $H = H_0^1(\Omega)$, $f = I_\lambda, \beta = 1 N S_B^{N/2} \|W\|^2_{x^{N/2}}, V = H_1, W = H_2, \xi = \xi_\lambda, \rho = \rho_\lambda, \beta' = \beta_\lambda$ and so, for

$$\lambda_+ - \lambda < \left(\frac{\eta}{\|W\|_{x^{N/2}}}\right)^{2N} S_B \left[\text{meas}(\Omega)\right]^{-2/N},$$
the problem \((P')\) admits at least
\[
m = \dim (H_1 \cap H_2) - \text{codim} (H_1 + H_2) = \dim M \left( \hat{\lambda}_+ \right)
\]
pairs of nontrivial solutions
\[
\{ u_k (\lambda), -u_k (\lambda) \}, \ k = 1, 2, ..., m.
\]

Since
\[
I_\lambda (u_k (\lambda)) \in [\delta, \delta'] \text{ and } \beta' \leq \frac{1}{N} \eta^{2-2N} \left( \hat{\lambda}_+ - \lambda \right)^{N/2} \left( \text{meas} (\Omega) \right) \to 0, \text{ as } \lambda \nearrow \hat{\lambda}_+,
\]
we obtain that
\[
I_\lambda (u_k (\lambda)) \to 0, \text{ as } \lambda \nearrow \hat{\lambda}_+, \ \forall k \in \{1, 2, ..., m\}.
\]

From this and from \(dI_\lambda (u_k (\lambda)) = 0\), we obtain that
\[
u_k (\lambda) \to 0, \text{ strongly in } H^1_0 (\Omega), \text{ as } \lambda \nearrow \hat{\lambda}_+.
\]
(6)
since \(I_\lambda\) satisfies the (PS) condition in the interval
\[
\left( -\infty, 1 NS_B^{N/2} \| W \|^{N-2}_\infty \right).
\]

Now, from the equivalence between \((P')\) and \((P)\), it is easy to observe that if \(\hat{\lambda}_+ - \lambda < (\eta \| W \|_\infty)^{2N} S_B \left[ \text{meas} (\Omega) \right]^{-2/N}\), then \((P)\) admits at least \(m\) pairs of nontrivial solutions \(\{ (u_k (\lambda), v_k (\lambda)); (-u_k (\lambda), -v_k (\lambda)) \}\), \(k = 1, 2, ..., m\), where \(v_k (\lambda) = 1 kB (u_k (\lambda))\). Moreover, from (6) and the continuity of \(B\), we also obtain that
\[
u_k (\lambda) \to 0, \text{ strongly in } H^1_0 (\Omega), \text{ as } \lambda \nearrow \hat{\lambda}_+.
\] and this ends the proof.

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