An unknotting theorem for tori in $S^4$.

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Abstract

Let $T$ be a torus in $S^4$ and $T^*$ a projection of $T$. If the singular set $\Gamma(T^*)$ consists of one disjoint simple closed curve, then $T$ can be moved to the standard position by an ambient isotopy of $S^4$.

1 Introduction

In this paper we will study an embedded torus $T$ in $S^4$. If the singular set of the projection $T^* (\subset S^3)$ of $T$ consists of one double curve, then what can be said about the position of $T$? The following theorem is the main result.

Main Theorem (Theorem 4.1). Let $T$ be a torus in $S^4$. If the singular set $\Gamma(T^*)$ consists of one simple closed curve, then $T$ can be moved to the standard position by an ambient isotopy of $S^4$.

We will work in the PL category. All submanifolds are assumed to be locally flat. Let $S^4$ be the 4-dimensional sphere, $S^3$ the 3-dimensional sphere, and $p : S^4 \setminus \{\infty\} \rightarrow S^3 \setminus \{\infty\}$ the projection defined by $p(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3)$.

Let $B = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_1^2 + x_2^2 + x_3^2 \leq 1\}$, and $P = B \cap \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_4 = 0\}$. Let $F$ be a closed oriented surface, and $f : F \rightarrow S^3 \setminus \{\infty\}$ a map. We say that $f$ is in general position, if for each element $x$ of $f(F)$, there exist a regular neighborhood $N$ of $x$ in $S^3 \setminus \{\infty\}$ and a homeomorphism $h : N \rightarrow B$ such that $N$ and $h$ satisfy the following two conditions:

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(1) Under $h$, $(N, N \cap f(F), z)$ is homeomorphic to either $(B, P_1, (0, 0, 0))$, $(B, P_1 \cup P_2, (0, 0, 0))$ or $(B, P_1 \cup P_2 \cup P_3, (0, 0, 0))$.

(2) Let $R$ be a component of $f^{-1}(f(F) \cap N)$. There exists an integer $i$ such that $h \circ f|_R : R \longrightarrow P_i$ is a homeomorphism.

**Note.** If $(N, N \cap f(F), z)$ is homeomorphic to $(B, P_1 \cup P_2, (0, 0, 0))$, then $z$ is called a **double point**. If $(N, N \cap f(F), z)$ is homeomorphic to $(B, P_1 \cup P_2 \cup P_3, (0, 0, 0))$, then $z$ is called a **triple point**.

Throughout this paper, we assume that $p|F$ is in general position.

With every point $P$ or subset $F$ of $S^4 \setminus \{\infty\}$, we associate the point $P^* = p(F)$ or the subset $F^* = p(F)$. We define $\Gamma(F^*)$ to be the set of all double points and triple points and put $\Gamma(F) = p^{-1}(\Gamma(F^*)) \cap F$.

A solid torus $V$ is said to be **standard** in $S^3$, if $V$ is a regular neighborhood of a trivial knot in $S^3$. And the torus $\partial V \subset S^3 \subset S^4$ is said to be a **standard torus** in $S^4$. In [H-K], they proved that a boundary of a handlebody in $S^4$ is unique up to ambient isotopies of $S^4$.

The circle is taken to be the quotient space $S^1 = \mathbb{R}/(\theta \sim \theta + 2\pi)$ for all $\theta \in \mathbb{R})$. We will write "$\theta \in S^1$". We denote by $(a, b)$ the greatest common divisor of the integers $a$ and $b$. Let $p_b : I \times S^1 \longrightarrow I \times S^1$ be the $b$-fold cyclic cover given by $(x, \theta) \mapsto (x, b\theta)$ for $b \in \mathbb{Z}\{0\}$. Let $R_\phi : I \times S^1 \longrightarrow I \times S^1$ be the rotation map given by $(x, \theta) \mapsto (x, \theta + \phi)$ for $\phi \in S^1$. Let $\alpha : S^1 \longrightarrow I \times S^1$ be an immersion. Let $i_\theta : I \times S^1 \longrightarrow I \times S^1 \times \theta \subset I \times S^1 \times S^1$ be the inclusion map $(x, \phi) \mapsto (x, \phi, \theta)$. Let $a, b$ be integers satisfying $b \neq 0$. We define immersed surfaces $\alpha(a, b)$ in $I \times S^1 \times S^1$, which satisfies

$$\alpha(a, b) \cap I \times S^1 \times \theta = i_\theta r_{a\theta/b}(p_b^{-1}(\alpha(S^1))).$$

In particular, we denote by $T_1(a, b)$ the immersed tori $\alpha(a, b)$ obtained from $\alpha$ shown in Figure 1.
All the homology groups are with coefficients in $\mathbb{Z}$.

**Example 1.1.** If $(a, b) = 1$ and $b \neq 0$, then there exists a torus $T$ in $S^4$ with $T^* = \alpha(a, b)$ (see [T, Theorem 8]).

**Example 1.2.** There exists an embedded torus $T'$ in $S^4$ with $T'^* = (a, b)$, where $(a, b) = 1, b \neq 0$. We can check that $(S^3, \Gamma(T'))$ is homeomorphic to $(S^3, (a, b)$-torus knot) where $(a, b)$-torus knot is defined in [R] (see p 53). Therefore $T_i(a, b)$ is the immersed torus having the singular set $\Gamma(T^*)$ of one simple closed curve.

## 2 Solid tori and immersed surfaces in $S^3$

**Lemma 2.1.** Let $V$ be a solid torus, $A$ a properly embedded annulus into $V$ with $[a_0] \neq 0$ in $H_1(V)$ where $a_0, a_1$ are the components of $\partial A$, then there exists an embedding map $h : A \times I \to V$ with $h(a, 0) = a$ for all $a \in A$, and $h(\partial A \times I \cup A \times 1) \subset \partial V$.

**Proof.** (Only outline). We find a disk $E$ such that $\partial E = l \cup k$, $l$ and $k$ are disjoint arcs, $\text{int} E \cap A = \phi$, $l \cap k = \partial l = \partial k$, $l \subset \partial V$, and $k \subset A$. Let $B$ be a component of $\partial V \setminus (a_0 \cup a_1)$ with $B \supset l$. Then $A \cup B$ is a torus. There exists a 3-manifold $W$ with $\partial W = A \cup B$, $W \supset E$. Let $N(E)$ be a regular neighborhood of $E$ in $W$. We have that $\partial N(E) = D_0 \cup C \cup D_1$ such that $D_i$ is a disk, $C$ is an annulus, and $\partial N(E) \cap \partial W = C$. Then $\partial(W \setminus N(E)) = (A \cup B \setminus C) \cup D_0 \cup D_1$ is a 2-sphere. By the Schönflies Theorem ([R] p 34), $W \setminus N(E)$ is a 3-ball. $W$ is obtained from $W \setminus N(E)$ by attaching a 1-handle $N(E)$. Therefore $W$ is a solid torus. We make a map $h$ by using $W$.

**Lemma 2.2.** If $V_1, V_2$, and $V_3$ are solid tori in $S^3$ such that $V_i \cap V_j = \partial V_i \cap \partial V_j$ is an annulus and $S^3 = V_1 \cup V_2 \cup V_3$, then there exist integers $i, j$ such that $V_i$ and $V_j$ are standard solid tori in $S^3$.

**Proof.** The set $V_1 \cap V_2 \cap V_3$ consists of two disjoint simple closed curves. Let $c$ be a component of $V_1 \cap V_2 \cap V_3$. We denote $c = p_i l_i + m_i \in H_1(\partial V_i)$ ($i=1, 2$ or $3$) where $l_i$ is a preferred longitude of $\partial V_i$, and $m_i$ is a meridian of
\( \partial V_i \), and \((p_i, q_i)\) is a pair of relatively prime integers. By van Kampen’s theorem, we have \( \pi_1(V_i \cup V_j) \cong \langle l_i, l_j \mid l_i^{p_i} = l_j^{p_j} \rangle \). We get

\[
H_1(V_i \cup V_j) \cong \begin{cases} 
\mathbb{Z} & \text{if } (p_i, p_j) = 1 \\
\mathbb{Z} \oplus \mathbb{Z} & \text{if } (p_i, p_j) = |d| \neq 1 \\
\mathbb{Z} \oplus \mathbb{Z}_{p_1} & p_k = 0, p_s \neq 0, \{k, s\} = \{i, j\} \\
\mathbb{Z} \oplus \mathbb{Z} & p_i = p_j = 0 
\end{cases}
\]

Since \( V_i \cup V_j \) is the complement of an open regular neighborhood of some knot, \( H_1(V_i \cup V_j) \cong \mathbb{Z} \). Hence we have to consider the following two cases:

1. \( p_i \neq 0, p_j \neq 0, (p_i, p_j) = 1 \) or
2. \( p_k = 0, p_s = \pm 1, \{k, s\} = \{i, j\} \).

Case (1).

We construct a Seifert fibration on \( S^3 \) in which each solid torus \( V_i \) has \( c \) as a fiber. If \( |p_i| \neq 1 \) for all \( i \), then there are three exceptional fibers. But we can show that in any Seifert fibration of the 3-sphere, there are at most two exceptional fibers (see [J-S] p 181). This is a contradiction. Hence there exists an integer \( k \) with \( p_k = \pm 1 \). We have \( \pi_1(V_i \cup V_k) \cong \langle l_i, l_k \mid l_i^{p_i} = l_k^{p_k} \rangle \cong \mathbb{Z} \). Therefore \( V_j \) is a standard solid torus \((j \neq i, k)\). Similarly, we can show that \( V_i \) is a standard solid torus.

Case (2).

Since \( c = q_k m_k = \pm l_k + q_s m_s \), we have \( q_k = \pm 1 \). There exists a disk \( D \) in \( V_k \) with \( c = \partial D \subset \partial V_k \). Hence \([c]=0\) in \( H_1(S^3 \setminus \text{int} V_s) \) and \( q_s = 0 \). The solid torus \( V_s \) is a regular neighborhood of some knot \( K \). But \( K \) is a boundary of some disk in \( S^3 \). Hence \( K \) is a trivial knot and \( V_s \) is a standard solid torus. Let \( V = V_k \cup V_s \). Since \( c = \pm m_k = \pm l_s \) and \( V_k \cap V_s \) is an annulus, then \( V \) is a solid torus. Let \( V_t \) be the third solid torus with \( t \neq k, s \). Then \( S^3 = V \cup V_t, V \cap V_t = \partial V = \partial V_t \). But up to homeomorphism there is only one way of decomposing \( S^3 \) into two solid tori with the same boundary. Therefore \( V_i \) is a standard solid torus.

Remark. Let \( V_i, V_j \) be as above. If \( H_1(V_i \cup V_j) \cong \mathbb{Z} \) and \([c]=0\) in \( H_1(V_i \cup V_j) \), then \( p_k = 0, p_s = \pm 1, \{k, s\} = \{i, j\} \).
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**Fact.** Let $F$ be a closed surface in $S^4$ with $p[F]$ in general position, and $c$ a simple closed curve in $S^3$ such that $c$ is transverse to $f(F)$, $c \cap \Gamma(F^*) = \phi$. Then the number of points of $c \cap \Gamma(F^*)$ is even.

**Lemma 2.3.** If $F$ is an oriented closed surface in $S^4$ with $p[F]$ in general position, then $F \setminus \Gamma(F)$ is divided into some regions. Then we can color each region black or white so that adjacent regions have different colors.

**Remark.** Suppose that $\Gamma(F^*)$ consists of double points, and let $n$ be a number of components in $\Gamma(F)$ which are not contractible in $F$. By Lemma 2.3, one sees that if $F$ is a torus, then $n$ is even.

**Proof.** Let $D_1, \ldots, D_s$ be the components of $S^3 \setminus F^*$. We will construct a function $f : \{D_1, \ldots, D_s\} \rightarrow \mathbb{Z}_2$. Let $x_0$ be a point of $S^3 \setminus F^*$, $x_i$ a point in $D_i$, and $l_i$ an arc in $S^3$ such that $l_i$ is transverse to $F^*$ and $\partial l_i = \{x_0, x_i\}$. We define $f(D_i) = 0$ if the number of points of $l_i \cap F^*$ is even, otherwise $f(D_i) = 1$. By Fact, we can show that $f$ does not depend choices of $x_i$ and $l_i$. And then $f$ satisfies the property that $D_i$ is an adjacent region of $D_j$ (i.e. there exists a path $l \subset S^3$ such that $l(0) \in D_i, l(1) \in D_j, l(1) \cap \Gamma(F^*) = \phi$, and $l(1) \cap F^* = \{\text{one point}\}$), then $f(D_i) \neq f(D_j)$. Let $E = \{E_1, \ldots, E_t\}$ be the components of $F^* \setminus \Gamma(F^*)$. The orientation of $F$ induces the orientation of $E_i$. We define a function $h : E \rightarrow \mathbb{Z}_2$ by $h(E_i) = 1$ if the positive normal vector of $E_i$ points to a white region, otherwise $h(E_i) = 0$. Using $h$, we color the regions of $F \setminus \Gamma(F)$.

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**Lemma 2.4.** Let $F$, $p[F]$ be as above, and $\gamma^*$ a component of $\Gamma(F^*)$. If $\gamma^*$ is a simple closed curve, then $p^{-1}(\gamma^*) \cap F$ consists of two disjoint simple closed curves.

**Proof.** Let $N$ be a regular neighborhood of $\gamma^*$ in $S^3$. Then $p^{-1}(N) \cap F$ consists of either two disjoint annuli, one M"obius band or two disjoint M"obius bands. Since $F$ is an oriented surface, $p^{-1}(N) \cap F$ consists of two disjoint annuli. Therefore $p^{-1}(\gamma^*) \cap F$ is two disjoint simple closed curves. This completes the proof of Lemma 2.4.
3 Local moves of surfaces in $S^4$

Lemma 3.1. Let $F$ be an oriented closed surface in $S^4$ with $p|F$ in general position. Let $\gamma^*$ be a component of $\Gamma(F^*)$ which is a simple closed curve, $c_1, c_2$ the components of $p^{-1}(\gamma^*) \cap F$. If $\gamma^*$ satisfies one of the following conditions, then $\gamma^*$ can be cancelled by an ambient isotopy of $S^4$.

1. There exist disks $D_1, D_2$ in $F$ with $\partial D_i = c_i$ and $\text{int} D_i \cap \Gamma(F) = \phi$.

2. There exists an annulus $A$ in $F$, and a solid torus $V$ in $S^3$ such that $\partial A = c_1 \cup c_2$, $\partial V = A^*$, $\text{int} V \cap F^* = \phi$, and $\gamma^*$ is a generator of $H_1(V) \cong \mathbb{Z}$.

3. There exists an annulus $A$ in $F$ with $\partial A = c_1 \cup c_2$, $[c_i] = 1$ in $\pi_1(F)$, and $\text{int} A \cap \Gamma(F) = \phi$.

Proof. If $\gamma^*$ satisfies (1), the lemma is proved by [Y, Lemma (4.4)]. If $\gamma^*$ satisfies (2), the proof is easy.

Suppose $\gamma^*$ satisfies (3). The surface $A^*$ is an embedded torus in $S^3$, and $\gamma^*$ is a simple closed curve on $A^*$. Since $[c_i] = 1$ in $\pi_1(F)$, there exist disks $D_i$ in $F$ with $\partial D_i = c_i$ (see [E, Theorem 1.7]). Let $D = D_i$ with $A \cap D_i = c_i$. Let $V_1, V_2$ be the closures of the components of $S^3 \setminus A^*$ with $V_1 \cup V_2 = S^3$, $\partial V_1 = A^*$, and $V_1 \supset F^* \cup D^*$. By the solid torus theorem (see [R] p107), either $V_1$ or $V_2$ is a solid torus. In general, $D^*$ is an immersed disk. By Dehn’s lemma, there exists a non-singular disk $E$ with $\text{int} E \cap A^* = \phi$ and $\partial E = \gamma^*$.

Case 1) $V_1$ is a solid torus.

Move $T$ by an ambient isotopy of $S^4$, then we may assume that $V_1$ is a standard solid torus. And $V_2$ is a standard solid torus, too. We have $\gamma^* = \partial E \subset \partial V_1$, $E \subset V_1$. Then $\gamma^*$ is a meridian of $V_1$ and a preferred longitude of $V_2$. We have $\partial A = c_1 \cup c_2$, $\partial V_2 = A^*$, $\text{int} V_2 \cap F^* = \phi$, and $[\gamma^*] = \pm 1$ in $H_1(V_2) \cong \mathbb{Z}$. Using Lemma 3.1 (2), we can prove the lemma in Case 1).

Case 2) $V_2$ is a solid torus.

Let $l$ be a preferred longitude of $\partial V_2$, $m$ a meridian of $\partial V_2$. We express $\gamma^* = pl + qm$ where $(p, q)$ is a pair of relatively prime integers. Since $\gamma^* = \partial E \subset \partial V_1$, then $E \subset V_1$ and $[\gamma^*] = 0$ in $H_1(V_1)$. Hence
We will define a symmetry-spun torus in $S^4$ (see [T]). Let $D^2 \times S^1$ be a solid torus, and $K$ a knot in $D^2 \times S^1$. Let $\tilde{p}_b : D^2 \times S^1 \rightarrow D^2 \times S^1$ be the $b$-fold cyclic cover given by $(x, \theta) \mapsto (x, b\theta)$ for $b \in \mathbb{Z}\{0\}$. Let $\tilde{r}_\phi : D^2 \times S^1 \rightarrow D^2 \times S^1$ be the rotation map given by $(x, \theta) \mapsto (x, \theta+\phi)$ for $\phi \in S^1$. Let $\tilde{r}_\theta : D^2 \times S^1 \rightarrow D^2 \times S^1 \times \theta \subset D^2 \times S^1 \times S^1$ be the inclusion map $(x, \phi) \mapsto (x, \phi, \theta)$. Let $a, b$ be integers satisfying $b \neq 0$. We define an embedded torus $T^a(K_b)$ in $D^2 \times S^1 \times S^1$, which satisfies

$$T^a(K_b) \cap D^2 \times S^1 \times \theta = \tilde{r}_\theta \tilde{r}_a \tilde{r}_b (\tilde{p}_b^{-1}(K)).$$

And we identify $D^2 \times S^1 \times S^1$ with a regular neighborhood of a standard torus in $S^4$. Then the torus $T^a(K_b)$ is called a symmetry-spun torus in $S^4$.

Let $T$ be a torus in $S^4$, $\alpha : S^1 \rightarrow I \times S^1$ an immersion. Suppose $T^a = \alpha(a, b)$ where $(a, b) = 1$, and $b \neq 0$. Then there exists a knot $\tilde{\alpha}$ in $D^2 \times S^1$ such that $T$ is ambient isotopic to $T^a(\tilde{\alpha})$.

**Remark.** Let $T$ be as above. There exists a symmetry-spun torus $T^a(\tilde{\alpha})$ in $S^4$ such that $(T^a(\tilde{\alpha}))^* = \alpha(a, b)$ and $T$ is ambient isotopic to $T^a(\tilde{\alpha})$.

**Lemma 3.2.** Let $T$ be a torus in $S^4$, and $\alpha$ an immersion from $S^1$ to $I \times S^1$ with $T^a = \alpha(a, b)$ where $(a, b) = 1$, and $b \neq 0$. Let $\tilde{\alpha}$ be a knot in $D^2 \times S^1$ obtained from as above. If $\tilde{\alpha}$ is a trivial knot in $S^3$, then $T$ can be moved to the standard position by an ambient isotopy of $S^4$.

**Proof.** We may assume that $T$ is ambient isotopic to $T^a(\tilde{\alpha})$. By [T,Theorem 8], then there exists a homeomorphism $f : S^4 \rightarrow S^4$ with $f(T^a(\tilde{\alpha})) = T^0(\tilde{\alpha}_1)$ or $T^1(\tilde{\alpha}_1)$. We easily check that $T^0(\tilde{\alpha}_1)$ and $T^1(\tilde{\alpha}_1)$ can be moved to the standard position by an ambient isotopy of $S^4$. Then there exists a solid torus $V$ in $S^4$ with $\partial V = T^0(\tilde{\alpha}_1)$ or $T^1(\tilde{\alpha}_1)$. Hence $\partial f^{-1}(V) = T^a(\tilde{\alpha}_b)$, and $f^{-1}(V)$ is a solid torus. By [H-K, Theorem 1.7], $T^a(\tilde{\alpha}_b)$ can be moved to the standard position by an ambient isotopy of $S^4$. 

$\blacksquare$
4 Main Theorem

Theorem 4.1. Let $T$ be a torus in $S^4$ with $p|T$ in general position. If $\Gamma(T^*)$ consists of one simple closed curve, then $T$ can be moved to the standard position by an ambient isotopy of $S^4$.

Proof. We distinguish four cases according to the position of $\Gamma(T)$. See Figure 2.

If the position of $\Gamma(T)$ is either I or II, then $T$ can be moved to the standard position by Lemma 3.1. The case III cannot happen by Lemma 2.3. We will consider the case IV. Let $A_1, A_2$ be the closures of the components of $T \setminus \Gamma(T)$, and $\gamma^* = \Gamma(T^*)$. Then $T_i = p(A_i)$ is an embedded torus, and $T_1 \cap T_2 = \gamma^*$. By the solid torus theorem, there exist solid tori $V_1, V_2$ with $\partial V_i = T_i$. We distinguish two cases: (1) $T_i \subset V_j$ or (2) $V_i \cap T_j = \gamma^* (\{i,j\} = \{1,2\})$.

Case 1) $T_1 \subset V_2$ or $T_2 \subset V_1$.

We may assume $T_1 \subset V_2$. Move $T$ by an ambient isotopy of $S^4$, and we suppose that $V_2$ is a standard solid torus.

(1-i) $[\gamma^*] = 0$ in $H_1(V_2)$.

The simple closed curve $\gamma^*$ is a meridian of $V_2$. Let $V = S^3 \setminus \text{int}V_2$. Then $A_2$ is an annulus satisfying $\partial A_2 = c_1 \cup c_2$, $\partial V = A_2^*$, $\text{int}V \cap F^* = \emptyset$, and $[\gamma^*]$ is a generator of $H_1(V) \cong \mathbb{Z}$. By Lemma 3.1 (2), $\gamma^*$ can be cancelled.

(1-ii) $[\gamma^*] \neq 0$ in $H_1(V_2)$.

Let $N$ be a regular neighborhood of $\gamma^*$ in $V_2$, $A = \text{cl}(\partial N \cap \text{int}V_2)$, and $a_0, a_1$ the components of $\partial A$. Then $A$ is an annulus, and $[a_i] \neq 0$ in $H_1(V_2)$. Cut $V_2$ by a meridian disk. We obtain Figure 3 (1) by Lemma
2.1. In Figure 3 the curve $\gamma^*$ is coiled four times to a preferred longitude of $V_2$. Let $V = V_2 \setminus N$, and $B = T_1 \setminus \text{int}N$. Then $V$ is a solid torus, and $B$ is an annulus. Let $b_0, b_1$ be the components of $\partial B$, then $[b_i] \neq 0$ in $H_1(V)$. We obtain Figure 3 (2) or (3) by Lemma 2.1. By Lemma 3.1 (2), we cancel $\gamma^*$ of Figure 3 (2). We see in Figure 3 (3) that $T^*$ is an immersed torus $T_1(a, b)$ with $(a, b) = 1, b \neq 0$. By Lemma 3.2, $T$ can be moved to the standard position. We completed the proof in Case 1).

Case 2) $V_1 \cap T_2 = \gamma^*$ or $V_2 \cap T_1 = \gamma^*$.

If $V_2 \supset V_1$ or $V_1 \supset V_2$, then we can use the method of Case 1). Therefore, we may assume $V_1 \cap V_2 = \gamma^*$. Let $N$ be a regular neighborhood of $\gamma^*$ in $S^3$, and $W = V_1 \cup N \cup V_2$. Then $\partial W$ is a torus.

(2-i) $[\gamma^*] = 0$ in $H_1(W)$.

We denote $\gamma^* = p_i l_i + q_i m_i \in H_1(\partial V_i)$ where $l_i$ is a preferred longitude of $\partial V_i$ and $m_i$ is a meridian of $\partial V_i$. We calculate $H_1(V_1 \cup V_2)$ in a similar way to Lemma 2.2. Since $H_1(W) \cong \mathbb{Z}$ and $[\gamma^*] = 0$ in $H_1(W)$, we have $p_i = 0, |q_i| = 1$ where $\{i, j\} = \{1, 2\}$ (see Remark after Lemma 2.2). Moreover, we get $|q_i| = 1$, and $\gamma^* = \pm l_i + q_i m_i$. Since $\gamma^*$ is a boundary of a meridional disk of $\partial V_i$, $V_i$ is a standard solid torus and $\gamma^* = \pm l_i$. By Lemma 3.1 (2), $\gamma^*$ can be cancelled.

(2-ii) $[\gamma^*] \neq 0$ in $H_1(W)$.

Suppose that $W$ is a solid torus. Let $A_i = V_i \cap \partial N$, and $a_0^i, a_1^i$ be the components of $\partial A_i$. Then $A_i$ is an annulus, and $[a_1^i] \neq 0$ in $H_1(W)$. Cut $W$ by a meridional disk $D$. Using Lemma 2.1, we get Figure 4 (1). Drawing the picture of $T^* \cap N \cap D$, then we get Figure 4 (2). Then we see $T^* \cap D$ in Figure 4 (3). Moreover, $\gamma^*$ satisfies Lemma 3.1 (2). Thus $\gamma^*$ can be cancelled.

Suppose that $W$ is not a solid torus. Let $V = S^3 \setminus \text{int}W$. By the solid torus theorem, $V$ is a solid torus. We find an annulus $A$ with $N \supset A \supset \gamma^*$, $\partial N \supset \partial A$, $A \cap (V_1 \cup V_2) = \gamma^*$, and $a_1 \subset J_1$ where $J_1$ and $J_2$ are components of $\partial N \setminus (\text{int}V_1 \cup \text{int}V_2)$ and $a_1, a_2$ are the components of $\partial A$. Let $N_i$ be the closure of the component of $N \setminus A$ with $N_i \cap \text{int}V_i \neq \emptyset$. Then $V_i \cup N_i$ is a solid torus. Let $Z_1 = V_1 \cup N_1$, $Z_2 = V_2 \cup N_2$ and $Z_3 = V$. Then $Z_i$ is a solid torus, $Z_i \cap Z_j = \partial Z_i \cap \partial Z_j$ is the annulus, and $S^3 = Z_1 \cup Z_2 \cup Z_3$. By Lemma 2.2 and the fact that $W$ is not a solid torus, we have that $Z_1$ and $Z_2$ are standard tori. Let $W_1 = V_1$, and $W_2 = S^3 \setminus \text{int}V_2$. Then $W_i$ is a solid torus, $\partial W_i = \partial V_i = T_i$, and $W_2 \supset W_1$. We can reduce the argument to Case 1).
This completes the proof of Theorem 4.1.

Figure 3

Figure 4

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