On pseudo-isotopy classes of homeomorphisms of \( \#_p (S^1 \times S^n) \).

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Abstract

We study self-homotopy equivalences and diffeomorphisms of the \((n + 1)\)-dimensional manifold \( X = \#_p (S^1 \times S^n) \) for any \( n \geq 3 \). Then we completely determine the group of pseudo-isotopy classes of homeomorphisms of \( X \) and extend to dimension \( n \) well-known theorems due to F. Laudenbach and V. Poenaru [10],[12] and J.M. Montesinos [14].

1 Introduction

Through the paper we work in the piecewise-linear (resp. \( C^\infty \)-differentiable) category, so we shall omit the prefix PL (resp. DIFF). Therefore the term homeomorphism means either PL homeomorphism or diffeomorphism.

Let \( M^{n+1} \) be a closed connected oriented \((n + 1)\)-manifold. Following [3] , [19], we say that two homeomorphisms \( f, g : M \to M \) are pseudo-isotopic if there is a homeomorphism \( F : M \times I \to M \times I \ (I = [0, 1]) \) such that \( F(x, 0) = f(x) \) and \( F(x, 1) = g(x) \) for all \( x \in M \).

Let us consider the following groups:

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\( \text{Aut}(M) \) (resp. \( \text{Aut}_0(M) \)) the group of (resp. orientation-preserving) self-homeomorphisms of \( M \);
\( \mathcal{D}(M) \) (resp. \( \mathcal{D}_0(M) \)) the group of pseudo-isotopy classes of (resp. orientation-preserving) homeomorphisms of \( M \);
\( \mathcal{E}(M) \) (resp. \( \mathcal{E}_0(M) \)) the group of homotopy classes of (resp. orientation-preserving) homotopy self-equivalences of \( M \);
\( \text{Aut}(\Pi_1) \) the group of automorphisms of the fundamental group \( \Pi_1 = \Pi_1(M) \) of \( M \);
\( \text{Out}(\Pi_1) \) the outer automorphism group of \( \Pi_1 \), i.e. automorphisms modulo inner automorphisms.

We have natural maps (base points are not required to be fixed)
\[ \text{Aut}(M) \to \mathcal{D}(M) \to \mathcal{E}(M) \to \text{Out}(\Pi_1) \]
\[ \text{Aut}_0(M) \to \mathcal{D}_0(M) \to \mathcal{E}_0(M) \to \text{Out}(\Pi_1). \]

In [3], [7], [9] it was studied the pseudo-isotopy classes of homeomorphisms (and self-equivalences) of the manifold \( M^{n+1} = S^1 \times S^n \) for \( n \geq 2 \). There it was shown that two homeomorphisms of \( S^1 \times S^n \) are homotopic if and only if they are pseudo-isotopic (resp. isotopic for the case \( n = 2 \)). Hence the natural map
\[ \mathcal{D}
\left(S^1 \times S^n\right) \to \mathcal{E}
\left(S^1 \times S^n\right) \]
is an isomorphism for any \( n \geq 2 \).

We summarize the results proved in the quoted papers by the following statement.

**Theorem 1.** ([3],[7],[9])

If \( n \geq 2 \), then
\[ \mathcal{D}
\left(S^1 \times S^n\right) \cong \mathcal{E}
\left(S^1 \times S^n\right) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \]

By Theorem 1, it follows that there are at most two non equivalent \( n \)-knots in the \((n + 2)\)-sphere with diffeomorphic complements, \( n \geq 2 \) (see [3], [7], [9]).

The aim of our paper is to extend Theorem 1 for the \((n + 1)\)-dimensional manifold \( X = \#_p (S^1 \times S^n) \), \( n \geq 2 \), \( p \geq 1 \), i.e. the connected sum of \( p \) copies of \( S^1 \times S^n \).
More precisely, we prove the following result

**Theorem 2.** If \( X = \#_p \left( S^1 \times S^n \right) \), \( n \geq 2, \ p \geq 1 \), then we have short exact sequences

\[
0 \rightarrow \bigoplus_{p+1} \mathbb{Z}_2 \rightarrow \mathcal{D}(X) \rightarrow \text{Out}(\Pi_1) \rightarrow 0,
\]

\[
0 \rightarrow \bigoplus_p \mathbb{Z}_2 \rightarrow \mathcal{D}_0(X) \rightarrow \text{Out}(\Pi_1) \rightarrow 0,
\]

where \( \Pi_1 = \Pi_1(X) \simeq \frac{\mathbb{Z}}{p} \) is the free group with \( p \) generators, \( p \geq 1 \).

Observe that the group \( \mathcal{D}(X) \) (resp. \( \mathcal{D}_0(X) \)) is not a direct sum of the other two terms of the sequence for \( p > 1 \). Indeed, diffeomorphisms of \( X \), which permute the \( p \) summands \( S^1 \times S^n \), also permute the \( p \) rotations along \( n \)-spheres (compare section 4).

As a consequence of Theorem 2, we completely determine the group \( \mathcal{D}_0(X) \) of \( X \) as follows:

**Theorem 3.** If \( X = \#_p \left( S^1 \times S^n \right) \), \( n \geq 2, \ p \geq 1 \), then the group \( \mathcal{D}_0(X) \simeq \mathcal{E}_0(X) \) is generated by sliding 1-handles, twisting 1-handles, permuting 1-handles and rotations.

The case \( n = 2 \) in the statement of Theorem 3 was proved by F. Laudenbach (see [11]) and J.M. Montesinos (see [14]). The definitions of the above generators can be found in [10] and [12]. Because all these generators extend to the \( (n+2) \)-handlebody \( Y = \#_p \left( S^1 \times D^{n+1} \right) \), i.e. the boundary connected sum of \( p \) copies of \( S^1 \times D^{n+1} \), we prove, following [14], other two consequences of Theorem 3 about handle presentations of manifolds.

**Corollary 4.** Let \( Y \) be the handlebody \( \#_p \left( S^1 \times D^{n+1} \right) \) with boundary \( \partial Y = X = \#_p \left( S^1 \times S^n \right), \ n \geq 2, \ p \geq 1 \). Given a connected compact \( (n+2) \)-manifold \( N^{n+2} \) with boundary \( \partial N \simeq X \), the smooth closed \( (n+2) \)-manifold \( M = N \cup_h Y \) obtained by gluing \( N \) and \( Y \) via an arbitrarily chosen diffeomorphism \( h : \partial N \to \partial Y \) is independent of the way of pasting the boundaries together.

In particular, the closed \( (n+2) \)-manifold \( M = Y \cup_h Y \) is diffeomorphic to the \( (n+2) \)-sphere \( S^{n+2} \).
Corollary 5. Each closed orientable \((n + 2)\)-manifold \(M^{n+2}\), \(n \geq 2\), with handle presentation

\[ M^{n+2} = H^0 \cup \lambda_1 H^1 \cup \ldots \cup \lambda_{n+1} H^{n+1} \cup H^{n+2} \]

is completely determined by

\[ H^0 \cup \lambda_1 H^1 \cup \ldots \cup \lambda_n H^n. \]

Here \(H^i\) represents an arbitrary handle of index \(i\).

Using Corollary 4, we prove an extension to dimension \(n\) of a well-known result due to F. Laudenbach and V. Poenaru (see [12]).

Corollary 6. Let \(M^{n+2}\) be the smooth closed \((n + 2)\)-manifold, \(n \geq 2\), obtained by gluing \(\#_p (S^1 \times D^{n+1})\) to \(\#_p (S^n \times D^2)\), \(p \geq 1\), via an arbitrary diffeomorphism of their boundaries. Then \(M\) is diffeomorphic to \(S^{n+2}\).

Proof. Set \(Y = \#_p (S^1 \times D^{n+1})\) and \(Z = \#_p (S^n \times D^2)\) for \(n \geq 2\) and \(p \geq 1\).

Consider a diffeomorphism \(h : \partial Y \rightarrow \partial Z\) and the smooth closed \((n + 2)\)-manifold \(M = Y \cup_h Z\).

One has canonical identifications

\[ \partial Y \xrightarrow{\alpha \cdot} X = \#_p (S^1 \times S^n) \xrightarrow{\beta \cdot} \partial Z \]

which will be given, one for all. It is obvious that \(Y \cup_{\beta} Z = S^{n+2}\).

Since the manifold \(M = Y \cup_h Z\) is independent of the way of pasting the boundaries together (see Corollary 4), it follows that \(M = Y \cup_h Z\) is diffeomorphic to \(Y \cup_{\beta} Z = S^{n+2}\).

\[ \blacksquare \]

2 Homotopy equivalences and pseudo-isotopies of \(X = \#_p (S^1 \times S^n)\)

In this section we prove that the group \(\mathcal{D}(X)\) of pseudo-isotopy classes of homeomorphisms of \(X = \#_p (S^1 \times S^n)\), \(n \geq 3\), is isomorphic to \(\mathcal{E}(X)\). For this, we use the following results proved in [4] and [5].
Theorem 7. Let $M^{n+1}$, $n \geq 4$, be a closed connected PL $(n + 1)$-manifold of the same homotopy type as $X = \#_p \left( S^1 \times S^n \right)$. Then $M$ is PL homeomorphic to $X$.

Theorem 8. Any homotopy self-equivalence of $X = \#_p \left( S^1 \times S^n \right)$, $n \geq 3$, is homotopic to a PL homeomorphism.

Theorem 7 extends the analogous result proved in [9] for $p = 1$ and Theorem 8 represents an extension of Lemma 16.2 of [18], $p = 1$ and $n = 3$.

In order to prove our result we need the following proposition.

Proposition 9. If $X = \#_p \left( S^1 \times S^n \right)$, $n \geq 3$, $p \geq 1$, then any PL homeomorphism $f : X \to X$, which is homotopic to the identity, is pseudo-isotopic to the identity.

Proof. Let $Y$ be the $(n + 2)$-handlebody, i.e. $Y$ is the boundary connected sum $Y = \#_p \left( S^1 \times D^{n+1} \right)$. Obviously we have $\partial Y = X$. As shown in [4], Proposition 3.1, the homeomorphism $f : X \to X$ extends over $Y$. To make the reading clear, we sketch the construction and refer to [4] for more details.

Form the closed $(n + 2)$-manifolds $M = Y \cup_f Y$ and $N = Y \cup_f Y$. Obviously $M$ is PL homeomorphic to $\#_p \left( S^1 \times S^{n+1} \right)$. Furthermore $N$ is homotopy equivalent to $M$ since $f$ is homotopic to the identity.

Let $i_1 : Y \to M$ and $j_1 : Y \to N$ (resp. $i_2 : Y \to M$ and $j_2 : Y \to N$) be the canonical inclusions of $Y$ into the first (resp. second) copy of it. For simplicity we identify $Y = i_1(Y) \subset M$ with $Y = j_1(Y) \subset N$ so that $M \cap N = Y$.

Note that

$$f = (j_2|_X)^{-1} \circ j_1|_X.$$  

Because $n \geq 3$, Theorem 7 implies that there is a PL homeomorphism

$$h : M^{n+2} \to N^{n+2}.$$  

By the tubular neighborhood theorem and the Whitney embedding theorem we may assume that $h$ is the identity on the first summand $Y = i_1(Y)$. Then the restriction of $h$ to the second copy $i_2(Y)$ of $Y$ in $M$ provides the required extension of the map $f$. Thus, let $g : Y \to Y$ be
a PL homeomorphism which extends $f$ to $Y$. One has the commutative diagram

\[
\begin{array}{ccc}
\Pi_1(X) & \xrightarrow{f_*} & \Pi_1(Y) \\
i_* \downarrow & & \downarrow i_* \\
\Pi_1(Y) & \xrightarrow{g_*} & \Pi_1(Y)
\end{array}
\]

where the inclusion-induced homomorphism $i_* : \Pi_1(X) \to \Pi_1(Y) \simeq \mathbb{Z}^p$ is bijective. Since $f_* = \text{identity}$, it follows that $g_* = \text{identity}$.

Let $S^1_i$ be the canonical $i$-th $S^1$-factor of $Y = \#_p \left( S^1 \times D^{n+1} \right)$ for $i = 1, 2, \ldots, p$. Then the 1-sphere $\Sigma^1 = g \left( S^1_i \right)$ is homotopic to $S^1_i$ because $g_* = \text{identity}$. Hence they are also isotopic as $\dim Y \geq 5$. Then we isotope $g$ to a map, also named $g$, which sends the 1-dimensional graph $G = \cup_{i=1}^p S^1_i$ (one-point union) in $Y$ to itself via the identity. Then we can also adjust the map $g$ so that it is the identity on a regular neighborhood of $G$ in $Y$. Moreover we may choose these isotopies keeping a collar of the boundary $X = \partial Y$ fixed. In other words, there exist two regular neighborhoods $V$ and $W$ of $G$ in $Y$ which satisfy the following properties:

1) $V \subset \text{int} W \subset \text{int} Y$
2) $g|_V = \text{identity}$
3) the previous isotopies are fixed outside $W$.

By the regular neighborhood collaring theorem (see [16], p. 36), the complement $Y \setminus \text{int} V$ can be identified with $X \times I$ where $\partial Y = X = X \times 0$ and $\partial V = X \times 1$ ($I = [0, 1]$). Then the restriction map

$g| : X \times I \to X \times I$

is a pseudo-isotopy between $g|_{X \times 0} = f$ and $g|_{X \times 1} = \text{identity}$ (use 2) above). Thus the homeomorphism $f : X \to X$ is pseudo-isotopic to the identity as claimed.

\[\square\]

**Corollary 10.** If $X = \#_p \left( S^1 \times S^n \right)$, $n \geq 3$, $p \geq 1$, then the natural map

$D(X) \to \mathcal{E}(X)$
is an isomorphism.

**Proof.** By Proposition 9 the map of the statement is injective. It is also surjective because each homotopy self-equivalence of $X$ is homotopic to a PL homeomorphism by Theorem 8.

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**Theorem 11.** If $X = \#_p \left( S^1 \times S^n \right)$, $n \geq 3$, $p \geq 1$, then we have the following exact sequence

$$0 \to \ker \theta_0 \cong \bigoplus_p \mathbb{Z}_2 \to D_0(X) \cong E_0(X) \xrightarrow{\theta_0} \text{Out}(\Pi_1) \to 0,$$

i.e. any two orientation-preserving diffeomorphisms $f, g : X \to X$ with

$$f_* = g_* : \Pi_1 \to \Pi_1$$

are pseudo-isotopic provided certain obstructions

$$\alpha_i \in \Pi_1 (\text{SO}(n + 1)) \cong \mathbb{Z}_2$$

vanish, $1 \leq i \leq p$.

In order to prove Theorem 11 we need the following lemma.

**Lemma 12.** Let $f, g : X \to X$ be two degree one maps.

If $f_* = g_* : \Pi_1 \to \Pi_1$, then $f_* = g_* : \Pi_q \to \Pi_q$ for all $q \leq n$.

**Proof.** We observe that $\Pi_i(X) = 0$ for $1 < i < n$, hence $f_* = g_* : \Pi_q \to \Pi_q$ for all $q < n$. By [12], p. 341, the Poincaré duality and the relation $\deg(f) = \deg(g) = 1$, we have the following commutative diagrams

$$
\begin{array}{cccc}
H_n(X; \mathbb{Z}) & \xrightarrow{\cong} & H^n(X; \mathbb{Z}) & \xrightarrow{\cong} & H^2(X; \mathbb{Z}) \\
\tilde{f}_* \downarrow \Phi & & \downarrow \Phi & & \downarrow \Phi \\
H_n(\tilde{X}; \mathbb{Z}) & \xrightarrow{\cong} & H^n(\tilde{X}; \mathbb{Z}) & \xrightarrow{\cong} & H^2(\tilde{X}; \mathbb{Z}) \\
\end{array}
$$

$$
\begin{array}{cccc}
H^1(X; \mathbb{Z}) & \xrightarrow{\cong} & H^1(X; \mathbb{Z}) & \xrightarrow{\cong} & H^1(\Pi_1; \mathbb{Z}) \\
\tilde{f}_* \downarrow \Phi & & \downarrow \Phi & & \downarrow \Phi \\
H^1(\tilde{X}; \mathbb{Z}) & \xrightarrow{\cong} & H^1(\tilde{X}; \mathbb{Z}) & \xrightarrow{\cong} & H^1(\Pi_1; \mathbb{Z}) \\
\end{array}
$$

where $\tilde{f}, \tilde{g} : \tilde{X} \to \tilde{X}$ are the liftings of $f, g$ respectively.
Since the hypothesis \( f_* = g_* : \Pi_1 \to \Pi_1 \) directly implies \( f_*^* = g_*^* \),
it follows that \( f_* = \tilde{g}_* : H_n(\tilde{X} ; \mathbb{Z}) \to H_n(\tilde{X} ; \mathbb{Z}) \). Then the Hurewicz isomorphism
\[
H_n(\tilde{X} ; \mathbb{Z}) \cong \Pi_n(\tilde{X}) \cong \Pi_n(X)
\]
implies that \( f_* = g_* : \Pi_n \to \Pi_n \) as required.

**Proof of Theorem 11.** Here we prove that the sequence

\[
\oplus_p \mathbb{Z}_2 \to \mathcal{D}_0(X) \to \text{Out}(\Pi_1) \to 0
\]
is exact. The injectivity of the term \( \oplus_p \mathbb{Z}_2 \) into \( \mathcal{D}_0(X) \) will follow from realizations of obstructions in section 4.

Suppose that \( f : X \to X = \#_p(S^1 \times S^n) \), \( n \geq 3 \), is an orientation-preserving diffeomorphism such that \( \theta_0(f) = 1 \). We can choose a representative (also named \( f \)) in the class of \( f \) which preserves the base point of \( X \) and \( f_* = \text{identity on } \Pi_1(X) \). Lemma 12 implies that \( f_* = \text{identity on } \Pi_q(X) \) for all \( q \leq n \). By Proposition 9 it is enough to show that \( f \) is homotopic (and hence pseudo-isotopic) to the identity \( \text{Id}_X : X \to X \).

We attempt to build up a homotopy \( F : X \times I \to X \) between \( f \) and \( \text{Id}_X \) in steps, using a filtration of \( X \) by subcomplexes.

Consider the handle presentation

\[
X = D^{n+1} \cup_{i=1}^p (D^1_i \times D^n_i) \cup_{j=1}^p (D^1_j \times D^n_j) \cup B^{n+1}
\]
where \( D, B \) are \((n+1)\)-cells and \( \chi, \psi \) are embeddings

\[
\chi : \bigcup_{i=1}^p (\partial D^1_i) \times D^n_i \to \partial D^{n+1} = S^n
\]

\[
\psi : \bigcup_{j=1}^p (\partial D^n_j) \times D^1_j \to \partial \left( D^{n+1} \cup_{i=1}^p (D^1_i \times D^n_i) \right)
\]

Our filtration starts with \( D^{n+1} \), then we successively add \( D^1_i \times 0, D^1_i \times D^n_i, D^n_j \times 0, D^n_j \times D^1_j \) and finally \( B^{n+1} \).

Now we regard \( f \) as a diffeomorphism of \( X \times 1 \) and seek to extend \( f \) on \( X \times 1 \) and the identity \( \text{Id} \) on \( X \times 0 \) to a map \( F : X \times I \to X \), where \( I = [0, 1] \).
Step 1. By the disc theorem \( f|_{D^{n+1}} \) and \( \text{Id}|_{D^{n+1}} \) are homotopic. Thus we choose a homotopy and define it

\[
F|_{D^{n+1} \times I} : D^{n+1} \times I \to D^{n+1} \subset X.
\]

Step 2. We next define \( F \) on \( D^1 \times I \). Now \( F|_{\partial(D^1 \times I)} \) is already given:
on \( \partial D^1 \times I \subset D^{n+1} \times I \) by Step 1, on \( D^1 \times 0 \) by the identity and on \( D^1 \times 1 \) by \( f \).
Because \( f_* = \text{identity} \) on \( \Pi_1(X) \), we can extend to some map

\[
D^1 \times I \to X.
\]

Indeed, let \( S^1 \) be the \( i \)-th \( S^1 \)-factor of \( X = \#_p (S^1 \times S^n) \), \( n \geq 3 \); the condition \( f_* = \text{identity} \) on \( \Pi_1 \) implies that the 1-sphere \( f(S^1) \) is homotopic to \( S^1 \) (and also isotopic as \( \text{dim} X \geq 4 \)).

Step 3. We now extend \( F \) to \( D^1 \times D^1 \times I \), i.e. to a tubular neighborhood of \( D^1 \times I \) in \( X \times I \). By the tubular neighborhood theorem it suffices to find a trivialisation of the normal bundle with the desired properties.
As in step 2 these turn out to be that a trivialisation is already given on the boundary \( \partial (D^1 \times I) \). The obstruction to extending this over \( D^1 \times I \) (since this is contractible, the bundle certainly is trivial) is an element (see [17])

\[
\alpha_i \in \Pi_1 (SO(n + 1)) \simeq Z_2
\]

(see [1], [8] for the stable homotopy of the orthogonal group, \( n \geq 3 \)).
If \( \alpha_i = 0 \), then the extension of the framing and hence of \( F \) is possible.

Step 4. We now assume that steps 1, 2, 3 have been successfully performed, i.e. \( F \) has been already defined on

\[
(D^{n} \times D^1 \times D^1) \times I.
\]

We next extend \( F \) on \( D^n \times I \). Now \( F|_{\partial(D^n \times I)} \) is already given:
on \( \partial D^n \times I \subset \partial (D^{n+1} \cup \bigcup_{i=1}^{p} (D^1 \times D^1)) \times I \) by step 2, on \( D^n \times 0 \) by the identity and on \( D^n \times 1 \) by \( f \).
Because \( f_\ast = \text{identity on } \Pi_n(X) \) (see Lemma 12) we can extend \( F \) to some map

\[
D^n_3 \times I \to X.
\]

Indeed, let \( S^m_j (j = 1, 2, \ldots, p) \) be the \( j \)-th \( S^m \)-factor of \( X \). The condition \( f_\ast = \text{identity on } \Pi_n(X) \) implies that the \( n \)-sphere \( f \left( S^m_j \right) \) is homotopic to \( S^m_j \).

**Step 5.** We have now to extend \( F \) to \( D^n_3 \times D^1_3 \times I \), i.e. to a tubular neighborhood of \( D^n_3 \times I \) in \( X \times I \). As remarked in step 3, the obstruction to extending a trivialisation given on the boundary \( \partial \left( D^n_3 \times I \right) \) to \( D^n_3 \times I \) is an element of \( \Pi_n(\text{SO}(2)) \simeq \Pi_n(S^2) \simeq 0 \) for \( n \geq 3 \). Thus the extension of the framing and hence of \( F \) is possible on the whole of \( (X \setminus \text{int}(B^{n+1})) \times I \). Then we complete the extension of \( F \) to \( X \times I \) by using the Alexander theorem.

Finally we prove that the homomorphism \( \theta_0 \) is surjective. Indeed, for any \( \xi \in \text{Out}(\Pi_1) \) there exists \( f \in \text{Aut}(X) \) such that \( f_\ast = \xi \) (see [12]). If \( \deg(f) = 1 \), then \( [f] \in D_0(X) \) and \( \theta_0[f] = \xi \). Otherwise we compose \( f \) with the homeomorphism

\[
r' = \#_p \left( \text{Id}_{S^1} \times r \right) : X \to X
\]

where \( r : S^n \to S^n \) is the reflection on the 1-st coordinate. Then \( [f \circ r'] \in D_0(X) \) and \( \theta_0[f \circ r'] = \xi \). Thus the proof is completed.

In section 4 we will show that any obstructions can be realized.

## 3 Alternative proofs

We can give an alternative proof of Theorem 11 by applying the classical obstruction theory (compare for example with [6]).

In order to do this, we need some algebraic lemmas which are interesting by itself.
Lemma 13.

1) Let $\Lambda = \mathbb{Z}[\Pi_1]$ be the group ring of $\Pi_1(X)$. Let $e_1, e_2, \ldots, e_p \in \Pi_1(X)$ be canonical generators and let

$$\sigma = (e_1 - 1, e_2 - 1, \ldots, e_p - 1) \in \bigoplus_p \Lambda.$$ 

Then the $\Lambda$-module $\Pi_n(X)$ is $\Lambda$-isomorphic to $(\bigoplus_p \Lambda)/\sigma \Lambda$.

2) The $\Lambda$-module $\Pi_{n+1}(X)$ is $\Lambda$-isomorphic to $(\bigoplus_{p-1} \Lambda^2) \oplus (\bigoplus \mathbb{Z}_2)$, where $\Lambda$ acts on $\mathbb{Z}_2$ via the map naturally induced by the augmentation $e : \Lambda \to \mathbb{Z}$.

Proof.

1) Let $X^{(q)}$ be the $q$-skeleton of the standard cellular decomposition

$$e^0 \cup pe^1 \cup pe^n \cup e^{n+1}$$

of $X$. Since $X^{(q)} = X^{(1)}$ for $1 \leq q < n$, we have

$$\Pi_n(X) \simeq \Pi_n(\tilde{X}) \simeq H_n(\tilde{X} ; \mathbb{Z}) \simeq H_n(X ; \Lambda)$$

and $H_n(X;\Lambda) \simeq H^1(X;\Lambda)$ by Poincaré duality. Here $\tilde{X}$ denotes the universal covering space of $X$.

To calculate $H^1(X;\Lambda)$ we consider the exact sequence

$$0 \to H_1(\tilde{X}^{(1)}, \tilde{X}^{(0)}) \to H_0(\tilde{X}^{(0)}) \to H_0(\tilde{X}^{(1)}) \to 0$$

which gives the following augmented $\Lambda$-chain complex

$$0 \to \text{Hom}_\Lambda(\mathbb{Z}, \Lambda) \xrightarrow{i^\#} \text{Hom}_\Lambda(\Lambda, \Lambda) \xrightarrow{\iota^\#} \text{Hom}_\Lambda(I(\Lambda), \Lambda) \to 0,$$

hence

$$H^1(X;\Lambda) \simeq \text{coker } i^\# \simeq \frac{\text{Hom}_\Lambda(I(\Lambda), \Lambda)}{\text{Im } i^\#}.$$

As $\Lambda$-module, $I(\Lambda)$ is isomorphic to

$$\Lambda(e_1 - 1) \oplus \Lambda(e_2 - 1) \oplus \ldots \oplus \Lambda(e_p - 1),$$
where
\[ e_1, e_2, \ldots, e_p \in \pi_1(X) \cong \pi_2 \mathbb{Z} \]
are canonical generators. If \( \varphi \in \text{Hom}_A(A, A) \), then \( i^\#(\varphi) \) corresponds to
\[ \sigma \varphi(1) \in \oplus_p \mathbb{Z} \cong \text{Hom}_A(I(A), A), \]
proving statement 1) of the lemma.

2) Let \( X^* \) be the CW-complex obtained from \( X = \#_p (S^1 \times S^n) \) by attaching \( p - 1 \) \((n + 1)\)-cells \( D^{n+1} \) along the \( n \)-spheres where the connected sum is taken. Observe that \( X^* \) is homotopy equivalent to the wedge \( \vee_p (S^1 \times S^n) \).

Furthermore one can easily verify the following isomorphisms:
\[ \pi_{n+1}(X^*) \cong \pi_{n+2}(X^*) \cong \oplus_p \mathbb{Z}_2 \]
\[ \pi_{n+1}(X^*, X) \cong \oplus \Lambda \quad \pi_{n+2}(X^*, X) \cong \oplus_\Lambda. \]
Thus the homotopy exact sequence of the pair \((X^*, X)\) yields
\[ \pi_{n+2}(X) \xrightarrow{j_*} \pi_{n+2}(X^*) \longrightarrow \pi_{n+2}(X^*, X) \]
\[ \longrightarrow \pi_{n+1}(X) \longrightarrow \pi_{n+1}(X^*) \longrightarrow 0. \]
Since \( j_* \) is an epimorphism, we obtain the result.

Given a \( \Lambda \)-module \( L \), we denote by \( H^*(X; L) \) the cohomology of the complex \( \text{Hom}_A(C_*(X) \otimes L) \), where \( C_*(X) = H_*(\tilde{X}^{(\ast)}, \tilde{X}^{(\ast-l)}) \).

Lemma 14.

1) \( H^n(X; \pi_n(X)) \cong \mathbb{Z} \)

2) \( H^{n+1}(X; \pi_{n+1}(X)) \cong \oplus_p \mathbb{Z}_2 \).
Proof.

1) By Poincaré duality, we have \( H^n(X; \Pi_n(X)) \cong H_1(X; \Pi_n(X)) \) (see [18]).

Using \( \Pi_n(X) \cong (\oplus_{p} \Lambda)/\sigma \Lambda \), one obtains the following exact sequence

\[
H_1(X; \Lambda) \rightarrow H_1(X; \oplus_{p} \Lambda) \rightarrow H_1(X; (\oplus_{p} \Lambda)/\sigma \Lambda)
\]

\[
\rightarrow \mathbb{Z} \otimes_{\Lambda} \Lambda \rightarrow \mathbb{Z} \otimes_{\Lambda} (\oplus_{p} \Lambda).
\]

Now \( H_1(X; \Lambda) \cong H_1(X; \oplus_{p} \Lambda) \cong 0 \) and \( \mathbb{Z} \otimes_{\Lambda} \Lambda \rightarrow \mathbb{Z} \otimes_{\Lambda} (\oplus_{p} \Lambda) \) is the null homomorphism because \( \sigma \) goes to zero. Hence we obtain

\[
H^n(X; \Pi_n(X)) \cong \mathbb{Z} \otimes_{\Lambda} \Lambda \cong \mathbb{Z}
\]
as claimed (use also [15], Theorem 1.12).

2) We have

\[
H^{n+1}(X; \Pi_{n+1}(X)) \cong_{PD} H_0(X; \Pi_{n+1}(X)) \cong \Pi_{n+1}(X)_{\Pi_1(X)}
\]

where \( \Pi_{n+1}(X)_{\Pi_1(X)} \) is the maximal quotient module of \( \Pi_{n+1}(X) \) (see [15], p. 266), i.e.

\[
\Pi_{n+1}(X)_{\Pi_1(X)} = \frac{\Pi_{n+1}(X)}{\{\lambda x : \lambda \in \Lambda, x \in \Pi_{n+1}(X)\}}.
\]

Because this quotient module is \( \Lambda \)-trivial (see [15]), Lemma 13 implies that

\[
H^{n+1}(X; \Pi_{n+1}(X)) \cong_{PD} \mathbb{Z}_2.
\]

Thus the proof is completed.

Theorem 11: second proof. Let \( f : X \rightarrow X = \#_p \left(S^1 \times S^n\right), n \geq 3, p \geq 1 \), be a homotopy self-equivalence of degree one such that \( \theta_0(f) = 1 \). As before, we can assume that \( f \) preserves the base point of \( X \) and that \( f_* = (\text{Id}_X)_* \) on \( \Pi_q(X) \) for all \( q \leq n \) (see Lemma 12). We have to study under that conditions \( f \) is homotopic to the identity \( \text{Id}_X \). We attempt to build up a homotopy \( h : X \times I \rightarrow X \) between \( f \) and \( \text{Id}_X \) in steps using a filtration of \( X \) by subcomplexes.
Let \( X(q) \) be the \( q \)-skeleton of the standard cellular decomposition
\[
X = e^0 \cup pe^1 \cup pe^n \cup e^{n+1}.
\]
Because \( f_* = (\text{Id}_X)_* \) on \( \Pi_1 \), there is a homotopy
\[
h : X^{(2)} \times I \to X
\]
between \( f|_{X^{(2)}} \) and \( \text{Id}_X|_{X^{(2)}} \).
The equalities \( X^{(q)} = X^{(1)} \) for \( 1 \le q < n \) imply that
\[
H^n(X; \Pi_q(X)) \simeq 0
\]
for all \( 1 \le q < n \). Thus the first obstruction lies in
\[
H^n(X; \Pi_n(X)) \simeq \mathbb{Z}
\]
(see Lemma 14). Let \( h : X^{(n-1)} \times I \to X \) be a homotopy between
\( f|_{X^{(n-1)}} \) and \( \text{Id}_X|_{X^{(n-1)}} \). The obstruction to extend \( h \) to \( X^{(n)} \) is the homotopy class of the map
\[
f \cup h \cup \text{Id}_X : X \times 0 \cup X^{(n-1)} \times I \cup X \times 1 \to X,
\]
i.e. for any \( i = 1, 2, \ldots, p \) we have
\[
\Delta_i(f, h, \text{Id}_X) = \left[ f \cup h \cup \text{Id}_X \mid e_i \times 0 \cup e_i \times I \cup e_i \times 1 \right] \in \Pi_n(X).
\]
In other words, the difference cochain is defined as follows:
\[
d(f, h, \text{Id}_X) : C_n(X) \to \Pi_n(X)
\]
\[
e^p \to \Delta_i(f, h, \text{Id}_X)
\]
hence the obstruction is the cohomology class
\[
|d(f, h, \text{Id}_X)| \equiv \left[ \sum_{i=1}^p \Delta_i \right] \in H^n(X; \Pi_n(X)) \simeq \mathbb{Z}.
\]
Let \( h' : X^{(n-1)} \times I \to X \) be a homotopy of \( \text{Id}|_{X^{(n-1)}} \) to \( \text{Id}|_{X^{(n-1)}} \). It is well-known that
\[
\Delta(f, h, \text{Id}_X) + \Delta(\text{Id}_X, h', \text{Id}_X) = \Delta(f, h + h', \text{Id}_X)
\]
where \( h + h' : X^{(n-1)} \times I \to X \) is defined by

\[
(h + h')(x, t) = \begin{cases} h(x, 2t) & 0 \leq t \leq \frac{1}{2} \\ h'(x, 2t - 1) & \frac{1}{2} \leq t \leq 1. \end{cases}
\]

Indeed, for each \( i = 1, 2, \ldots, p \), we take a small ball in the centre of the \( n \)-cell \( e^n_i \) and cut off it. Next we attach to its place a spheroid representing the value in \( \Pi_n(X) \) of the cochain \( d \) at \( e^n_i \). Thus we can always choose an \( h' \) such that

\[
d(\text{Id}_X, h', \text{Id}_X) = -d(f, h, \text{Id}_X),
\]

i.e. \( h + h' \) extends to a homotopy \( X^{(n)} \times I \to X \) between \( f|_{X^{(n)}} \) and \( \text{Id}_X|_{X^{(n)}} \). Now the only obstructions lie in

\[
H^{n+1}(X; \Pi_{n+1}(X)) \cong \bigoplus_p \mathbb{Z}_2.
\]

This proves Theorem 11.

4 Realizing obstructions

Now we are going to prove Theorem 3.

Let \( \{e_i\}, i = 1, 2, \ldots, p \), be a free basis of \( \Pi_1(X) \cong \#_p \mathbb{Z} \), where \( X = \#_p \left(S^1 \times S^n\right), p \geq 1, n \geq 3 \). Obviously \( e_i \) is the homotopy class of the \( i \)-th \( S^1 \)-factor \( S^1_i \) of \( X \). As proved in [10] and [12], \( \text{Aut}(\Pi_1) \) is generated by sliding 1-handles, twisting 1-handles and permuting 1-handles. More precisely, for \( i = 2, 3, \ldots, p \) \((p > 1)\) define \( \phi_i \in \text{Aut}(\Pi_1) \) by setting \( \phi_i(e_1) = e_i, \phi_i(e_i) = e_1 \) and \( \phi_i(e_j) = e_j \) for each \( j \neq i, j \neq 1 \). Permuting the 1-handles \( e_i \) and \( e_j \) corresponds to the automorphism \( \phi_i \circ \phi_j \circ \phi_i^{-1} \).

It follows that \( \phi_i^2 = 1 \) and by [10], [12] there exist diffeomorphisms \( f_i : X \to X \) (permuting 1-handles) such that \( f_i \circ \phi_i = \phi_i \). Then define \( \sigma \in \text{Aut}(\Pi_1) \) by setting \( \sigma(e_1) = e_1^{-1} \) and \( \sigma(e_i) = e_i \) for \( i \neq 1 \). Twisting the 1-handle \( e_i \) corresponds to the automorphism \( \phi_i \circ \sigma \circ \phi_i^{-1} \). Obviously \( \sigma^2 = 1 \). Furthermore there exist diffeomorphisms of \( X \) (twisting 1-handles) which realize \( \sigma \) and \( \phi_i \circ \sigma \circ \phi_i^{-1} \) for \( i \geq 2 \). Finally we define
ψ ∈ Aut(Π₁), p > 1, by setting ψ(e₁) = e₁e₂ and ψ(eᵢ) = eᵢ for i ≥ 2 (sliding 1-handles).

Let Σᵢ = Sⁿᵢ be the i-th Sⁿ-factor of \( X = \#ₚ \left( S¹ × Sⁿ \right) \), p ≥ 1, n ≥ 3. Following [10], we show that rotations of X parallel to \( Σᵢ \) generate the obstruction subgroup

\[
\text{Ker} \theta₀ \cong \bigoplusₚ \Piₙ \left( \text{SO}(n + 1) \right) \cong \bigoplusₚ \mathbb{Z}₂.
\]

Let

\[\alpha : (S¹, 1) \to (\text{SO}(n + 1), \text{id})\]

be a loop representing a homotopy class of \( \Piₙ \left( \text{SO}(n + 1) \right) \cong \mathbb{Z}₂ \) \( (n ≥ 3) \).

Then \( \alpha \) induces a diffeomorphism

\[h_\alpha : Sⁿ × I \to Sⁿ × I\]

defined by

\[h_\alpha(x, t) = (\alpha(t)x, t)\]

for all \( x ∈ Sⁿ \) and \( t ∈ I = [0, 1] \). Obviously \( h_\alpha \) is the identity on the boundary \( ∂(Sⁿ × I) = Sⁿ × \{0\} ∪ Sⁿ × \{1\} \).

Now let \( M^{n+1} \) be a closed oriented \((n + 1)\)-manifold and let \( Σⁿ \) be an oriented n-sphere embedded in \( M \). Suppose \( \varphi : Sⁿ × I \to M \) is an orientation-preserving embedding such that \( \varphi(Sⁿ × 0) = Σ \). Because \( h_\alpha = \text{identity on } ∂(Sⁿ × I) \), one obtains a diffeomorphism

\[h_\alpha^Σ : M \to M\]

defined by

\[h_\alpha^Σ(x) = \begin{cases} x & \text{if } x ∈ M \setminus \text{Im} \varphi \\ \varphi ∘ h ∘ ϕ⁻¹(x) & \text{if } x ∈ \text{Im} \varphi. \end{cases}\]

We call the diffeomorphism \( h_\alpha^Σ \) the \( \alpha \)-rotation of \( M \) parallel to \( Σ \) (briefly, a rotation). Obviously the pseudo-isotopy class of \( h_\alpha^Σ \) depends only on the homotopy (resp. isotopy) class of \( \alpha \) (resp. \( Σ \)).

If \( M^{n+1} = X = \#ₚ \left( S¹ × Sⁿ \right) \), p ≥ 1, n ≥ 3, then let \( Σᵢ = Sᵢⁿ \) be the i-th Sⁿ-factor of \( X \). We set

\[h_{i,\alpha} = h_\alpha^Σᵢ\]
for $i = 1, 2, \ldots, p$ and $[\alpha] \in \Pi_1(\text{SO}(n + 1)) \simeq \mathbb{Z}_2$. One can choose $h_{i, \alpha}$ to be the identity on the union $\cup_{i=1}^p \Sigma_i$. Because $(h_{i, \alpha})_* = \text{identity on } \Pi_q(X)$ for all $q \leq n$, we have that $h_{i, \alpha} \in \text{Ker } \theta_0$, $i = 1, 2, \ldots, p$. Moreover $h_{i, \alpha} \circ h_{j, \beta} = h_{j, \beta} \circ h_{i, \alpha}$ ($i \neq j$), each $h_{i, \alpha}$ commutes with the generators of $\text{Aut}(\Pi_1)$ and $h_{i, \alpha}$ is pseudo-isotopic to the identity if and only if $[\alpha] = 0$. Thus we have shown that the rotations $h_i = h_{i, \alpha}$ of $X$ parallel to the $n$-spheres $\Sigma_i$ generate $\text{Ker } \theta_0$ if $[\alpha]$ is the generator of $\Pi_1(\text{SO}(n + 1)) \simeq \mathbb{Z}_2$. In particular, this shows that the term $\oplus_p \mathbb{Z}_2$ injects into $D_0(X)$.

More precisely, we can interpret our results in the following way (which is related to Lemma 5.4 of [10]):

**Corollary 15.** Let $X = \#_p (S^1 \times S^n), p \geq 1$, $n \geq 3$, and let $f : X \to X$ be an orientation-preserving diffeomorphism such that $\theta_0(f) = 1$, i.e. $f_* = \text{identity on } \Pi_1$. Then there exist loops (obstructions)

$$
\alpha_i : (S^1, 1) \to (\text{SO}(n + 1), \text{Id})
$$

($i = 1, 2, \ldots, p$) such that $f$ is pseudo-isotopic to the product

$$
h_{1, \alpha_1} \circ h_{2, \alpha_2} \circ \ldots \circ h_{p, \alpha_p}.
$$

Moreover, the pseudo-isotopy can be chosen keeping the union $\cup_{i=1}^p \Sigma_i$ fixed.

In other words, rotations $h_i = h_{i, \alpha}$ ($i = 1, 2, \ldots, p$) is a free basis of

$$\text{Ker } \theta_0 \simeq \oplus_p \Pi_1(\text{SO}(n + 1)) \simeq \oplus_p \mathbb{Z}_2$$

where $[\alpha]$ is the generator of $\Pi_1(\text{SO}(n + 1)) \simeq \mathbb{Z}_2$.

**5 Concluding remarks**

Following [12], let $C(n, \lambda)$ denote the class of smooth $(n + 1)$-manifolds of the form

$$N^{n+1} = H^0 \cup \mu H^\lambda \cup \mu H^{\lambda+1}$$

such that $N$ is contractible, $n \geq 3, 1 \leq \lambda \leq n - 1$. The $h$-cobordism theorem of Smale implies that if $N \in C(n, \lambda)$, then $N$ is an $(n + 1)$-disc provided that $n \geq 5$ and $1 \leq \lambda \leq n - 3$ (see for example [16]). On
the other hand $C(n, n - 2)$ contains elements with non-simply connected boundary (see [12]).

Now one might ask the following question:

$$N \in C(n, n - 1) \implies N \overset{\text{diff}}{\sim} D^{n+1}.$$  

We can apply Corollary 6 to give a positive answer in the particular case

$$H^0 \cup \pi H^{n-1} \simeq \#_p \left( S^{n-1} \times D^2 \right).$$

Indeed, we have the following result.

**Proposition 16.** Let $N^{n+1}$ be the manifold obtained by attaching $p$ handles of index $n$ to $\#_p \left( S^{n-1} \times D^2 \right)$, $n \geq 3$. If $H_{n-1} (N; \mathbb{Z}) = 0$, then $\partial N$ is diffeomorphic to the $n$-sphere $S^n$.

**Proof.** We simply follow the proof of Lemma 5 [12], settled for $n = 3$.

First of all, the hypothesis $H_{n-1} (N; \mathbb{Z}) = 0$ implies that $N$ is contractible, i.e. $N \in C(n, n - 1)$.

Let $\psi_i^n$ be the attaching map of the $i$-th handle $H^n_i = D^n_i \times D^1_i$ of index $n$, $i = 1, 2, \ldots, p$. The same argument as in [12] shows that $\psi_i^n \left( \partial D^n_i \times \frac{1}{2} \right)$ are disjoined homologically independent $(n-1)$-spheres embedded in

$$\partial \left( \#_p \left( S^{n-1} \times D^2 \right) \right) = \#_p \left( S^{n-1} \times S^1 \right).$$

Let $\Sigma_i^{n-1}$ be the $i$-th $(n-1)$-factor of $\#_p \left( S^{n-1} \times S^1 \right)$. Cutting $\#_p \left( S^{n-1} \times S^1 \right)$ along the $(n-1)$-spheres $\psi_i^n \left( \partial D^n_i \times \frac{1}{2} \right)$ (resp. $\Sigma_i^{n-1}$) yields a punctured $n$-disc $P^n$ (resp. $Q^n$), where

$$P^n \simeq Q^n \simeq D^n \setminus \{ 2p - 1 \text{ open } n \text{-cells} \}.$$  

A diffeomorphism $P^n \to Q^n$ which preserves the boundary components induces a diffeomorphism between the pairs

$$\left( \#_p \left( S^{n-1} \times S^1 \right), \bigcup_i \psi_i^n \left( \partial D^n_i \times \frac{1}{2} \right) \right)$$

and
On pseudo-isotopy classes of...

\[ \#_p \left( S^{n-1} \times S^1 \right) \cup \bigcup_{i} \Sigma_i^{n-1}. \]

This implies the statement.

\[ \blacksquare \]

References


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