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On pseudo-isotopy classes of homeomorphisms of $\#_p\left(\mathbb{S}^1 \times \mathbb{S}^n\right)$.

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Abstract

We study self-homotopy equivalences and diffeomorphisms of the $(n+1)$-dimensional manifold $X = \#_p\left(\mathbb{S}^1 \times \mathbb{S}^n\right)$ for any $n \geq 3$. Then we completely determine the group of pseudo-isotopy classes of homeomorphisms of X and extend to dimension n well–known theorems due to F. Laudenbach and V. Poenaru [10],[12] and J.M. Montesinos [14].

1 Introduction

Through the paper we work in the piecewise–linear (resp. C^∞-differentiable) category, so we shall omit the prefix PL (resp. DIFF). Therefore the term *homeomorphism* means either PL homeomorphism or diffeomorphism.

Let M^{n+1} be a closed connected oriented $(n+1)$-manifold. Following [3] , [19], we say that two homeomorphisms $f, g : M \to M$ are *pseudo-isotopic* if there is a homeomorphism $F : M \times I \to M \times I$ $(I = [0,1])$ such that $F(x,0) = f(x)$ and $F(x,1) = g(x)$ for all $x \in M$.

Let us consider the following groups:

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$\mathrm{Aut}(M)$ (resp. $\mathrm{Aut}_0(M)$) the group of (resp. orientation-preserving) self-homeomorphisms of M;

$\mathcal{D}(M)$ (resp. $\mathcal{D}_0(M)$) the group of pseudo-isotopy classes of (resp. orientation-preserving) homeomorphisms of M;

$\mathcal{E}(M)$ (resp. $\mathcal{E}_0(M)$) the group of homotopy classes of (resp. orientation-preserving) homotopy self-equivalences of M;

$\mathrm{Aut}(\Pi_1)$ the group of automorphisms of the fundamental group $\Pi_1 = \Pi_1(M)$ of M;

$\mathrm{Out}(\Pi_1)$ the outer automorphism group of Π_1, i.e. automorphisms modulo inner automorphisms.

We have natural maps (base points are not required to be fixed)

$$\mathrm{Aut}(M) \to \mathcal{D}(M) \to \mathcal{E}(M) \to \mathrm{Out}(\Pi_1)$$

$$\mathrm{Aut}_0(M) \to \mathcal{D}_0(M) \to \mathcal{E}_0(M) \to \mathrm{Out}(\Pi_1).$$

In [3], [7], [9] it was studied the pseudo-isotopy classes of homeomorphisms (and self-equivalences) of the manifold $M^{n+1} = \boldsymbol{S}^1 \times \boldsymbol{S}^n$ for $n \geq 2$. There it was shown that two homeomorphisms of $\boldsymbol{S}^1 \times \boldsymbol{S}^n$ are homotopic if and only if they are pseudo-isotopic (resp. isotopic for the case $n = 2$). Hence the natural map

$$\mathcal{D}\left(\boldsymbol{S}^1 \times \boldsymbol{S}^n\right) \to \mathcal{E}\left(\boldsymbol{S}^1 \times \boldsymbol{S}^n\right)$$

is an isomorphism for any $n \geq 2$.

We summarize the results proved in the quoted papers by the following statement.

Theorem 1. ([3],[7],[9])
If $n \geq 2$, then

$$\mathcal{D}\left(\boldsymbol{S}^1 \times \boldsymbol{S}^n\right) \underset{iso}{\cong} \mathcal{E}\left(\boldsymbol{S}^1 \times \boldsymbol{S}^n\right) \underset{iso}{\cong} \boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2$$

By Theorem 1, it follows that there are at most two non equivalent n-knots in the $(n+2)$-sphere with diffeomorphic complements, $n \geq 2$ (see [3], [7], [9]).

The aim of our paper is to extend Theorem 1 for the $(n+1)$-dimensional manifold $X = \#_p\left(\boldsymbol{S}^1 \times \boldsymbol{S}^n\right)$, $n \geq 2$, $p \geq 1$, i.e. the connected sum of p copies of $\boldsymbol{S}^1 \times \boldsymbol{S}^n$.

More precisely, we prove the following result

Theorem 2. *If $X = \#_p \left(S^1 \times S^n\right)$, $n \geq 2$, $p \geq 1$, then we have short exact sequences*

$$0 \to \bigoplus_{p+1} \mathbb{Z}_2 \to \mathcal{D}(X) \to \mathrm{Out}(\Pi_1) \to 0,$$

$$0 \to \bigoplus_{p} \mathbb{Z}_2 \to \mathcal{D}_0(X) \to \mathrm{Out}(\Pi_1) \to 0,$$

where $\Pi_1 = \Pi_1(X) \simeq \mathop{*}_{p} \mathbb{Z}$ is the free group with p generators, $p \geq 1$.

Observe that the group $\mathcal{D}(X)$ (resp. $\mathcal{D}_0(X)$) is not a direct sum of the other two terms of the sequence for $p > 1$. Indeed, diffeomorphisms of X, which permute the p summands $S^1 \times S^n$, also permute the p rotations along n-spheres (compare section 4).

As a consequence of Theorem 2, we completely determine the group $\mathcal{D}_0(X)$ of X as follows:

Theorem 3. *If $X = \#_p \left(S^1 \times S^n\right)$, $n \geq 2$, $p \geq 1$, then the group $\mathcal{D}_0(X) \underset{iso}{\simeq} \mathcal{E}_0(X)$ is generated by sliding 1-handles, twisting 1-handles, permuting 1-handles and rotations.*

The case $n = 2$ in the statement of Theorem 3 was proved by F. Laudenbach (see [11]) and J.M. Montesinos (see [14]). The definitions of the above generators can be found in [10] and [12]. Because all these generators extend to the $(n+2)$-handlebody $Y = \#_p \left(S^1 \times D^{n+1}\right)$, i.e. the boundary connected sum of p copies of $S^1 \times D^{n+1}$, we prove, following [14], other two consequences of Theorem 3 about handle presentations of manifolds.

Corollary 4. *Let Y be the handlebody $\#_p \left(S^1 \times D^{n+1}\right)$ with boundary $\partial Y = X = \#_p \left(S^1 \times S^n\right)$, $n \geq 2$, $p \geq 1$. Given a connected compact $(n+2)$-manifold N^{n+2} with boundary $\partial N \simeq X$, the smooth closed $(n+2)$-manifold $M = N \cup_h Y$ obtained by gluing N and Y via an arbitrarily chosen diffeomorphism $h : \partial N \to \partial Y$ is independent of the way of pasting the boundaries together.*

In particular, the closed $(n+2)$-manifold $M = Y \cup_h Y$ is diffeomorphic to the $(n+2)$-sphere S^{n+2}.

Corollary 5. *Each closed orientable $(n+2)$-manifold M^{n+2}, $n \geq 2$, with handle presentation*

$$M^{n+2} = H^0 \cup \lambda_1 H^1 \cup \ldots \cup \lambda_{n+1} H^{n+1} \cup H^{n+2}$$

is completely determined by

$$H^0 \cup \lambda_1 H^1 \cup \ldots \cup \lambda_n H^n.$$

Here H^i represents an arbitrary handle of index i.

Using Corollary 4, we prove an extension to dimension n of a well-known result due to F. Laudenbach and V. Poenaru (see [12]).

Corollary 6. *Let M^{n+2} be the smooth closed $(n+2)$-manifold, $n \geq 2$, obtained by gluing $\#_p\left(S^1 \times D^{n+1}\right)$ to $\#_p\left(S^n \times D^2\right)$, $p \geq 1$, via an arbitrary diffeomorphism of their boundaries. Then M is diffeomorphic to S^{n+2}.*

Proof. Set $Y = \#_p\left(S^1 \times D^{n+1}\right)$ and $Z = \#_p\left(S^n \times D^2\right)$ for $n \geq 2$ and $p \geq 1$.

Consider a diffeomorphism $h : \partial Y \to \partial Z$ and the smooth closed $(n+2)$-manifold $M = Y \cup_h Z$.

One has canonical identifications

$$\partial Y \xrightarrow{\alpha} X = \#_p\left(S^1 \times S^n\right) \xleftarrow{\beta} \partial Z$$

which will be given, one for all. It is obvious that $Y \cup_{\beta^{-1}\circ\alpha} Z = S^{n+2}$.

Since the manifold $M = Y \cup_h Z$ is independent of the way of pasting the boundaries together (see Corollary 4), it follows that $M = Y \cup_h Z$ is diffeomorphic to $Y \cup_{\beta^{-1}\circ\alpha} Z = S^{n+2}$.

∎

2 Homotopy equivalences and pseudo-isotopies of $X = \#_p\left(S^1 \times S^n\right)$

In this section we prove that the group $\mathcal{D}(X)$ of pseudo-isotopy classes of homeomorphisms of $X = \#_p\left(S^1 \times S^n\right)$, $n \geq 3$, is isomorphic to $\mathcal{E}(X)$. For this, we use the following results proved in [4] and [5].

Theorem 7. *Let M^{n+1}, $n \geq 4$, be a closed connected PL $(n+1)$-manifold of the same homotopy type as $X = \#_p \left(S^1 \times S^n\right)$. Then M is PL homeomorphic to X.*

Theorem 8. *Any homotopy self-equivalence of $X = \#_p \left(S^1 \times S^n\right)$, $n \geq 3$, is homotopic to a PL homeomorphism.*

Theorem 7 extends the analogous result proved in [9] for $p = 1$ and Theorem 8 represents an extension of Lemma 16.2 of [18], $p = 1$ and $n = 3$.

In order to prove our result we need the following proposition.

Proposition 9. *If $X = \#_p \left(S^1 \times S^n\right)$, $n \geq 3$, $p \geq 1$, then any PL homeomorphism $f : X \to X$, which is homotopic to the identity, is pseudo-isotopic to the identity.*

Proof. Let Y be the $(n+2)$-handlebody, i.e. Y is the boundary connected sum $Y = \#_p \left(S^1 \times D^{n+1}\right)$. Obviously we have $\partial Y = X$. As shown in [4], Proposition 3.1, the homeomorphism $f : X \to X$ extends over Y. To make the reading clear, we skecth the construction and refer to [4] for more details.

Form the closed $(n+2)$-manifolds $M = Y \cup_{Id} Y$ and $N = Y \cup_f Y$. Obviously M is PL homeomorphic to $\#_p \left(S^1 \times S^{n+1}\right)$. Furthermore N is homotopy equivalent to M since f is homotopic to the identity.

Let $i_1 : Y \to M$ and $j_1 : Y \to N$ (resp. $i_2 : Y \to M$ and $j_2 : Y \to N$) be the canonical inclusions of Y into the first (resp. second) copy of it. For simplicity we identify $Y = i_1(Y) \subset M$ with $Y = j_1(Y) \subset N$ so that $M \cap N = Y$.

Note that

$$f = (j_2|_X)^{-1} \circ j_1|_X.$$

Because $n \geq 3$, Theorem 7 implies that there is a PL homeomorphism

$$h : M^{n+2} \to N^{n+2}.$$

By the tubular neighborhood theorem and the Whitney embedding theorem we may assume that h is the identity on the first summand $Y = i_1(Y)$. Then the restriction of h to the second copy $i_2(Y)$ of Y in M provides the required extension of the map f. Thus, let $g : Y \to Y$ be a PL homeomorphism which extends f to Y. One has the commutative diagram

$$\begin{array}{ccc} \Pi_1(X) & \xrightarrow{f_*} & \Pi_1(X) \\ i_* \downarrow & & \downarrow i_* \\ \Pi_1(Y) & \xrightarrow[g_*]{} & \Pi_1(Y) \end{array}$$

where the inclusion-induced homomorphism $i_* : \Pi_1(X) \to \Pi_1(Y) \simeq \underset{p}{*} \boldsymbol{Z}$ is bijective. Since $f_* =$ identity, it follows that $g_* =$ identity.

Let S_i^1 be the canonical i-th S^1-factor of $Y = \#_p \left(S^1 \times D^{n+1}\right)$ for $i = 1, 2, \ldots, p$. Then the 1-sphere $\Sigma_i^1 = g\left(S_i^1\right)$ is homotopic to S_i^1 because $g_* =$ identity. Hence they are also isotopic as $\dim Y \geq 5$. Then we isotope g to a map, also named g, which sends the 1-dimensional graph $G = \vee_{i=1}^p S_i^1$ (one-point union) in Y to itself via the identity. Then we can also adjust the map g so that it is the identity on a regular neighborhood of G in Y. Moreover we may choose these isotopies keeping a collar of the boundary $X = \partial Y$ fixed. In other words, there exist two regular neighborhoods V and W of G in Y which satisfy the following properties:

1) $V \subset \operatorname{int} W \subset \operatorname{int} Y$
2) $g|_V =$ identity
3) the previous isotopies are fixed outside W.

By the regular neighborhood collaring theorem (see [16], p. 36), the complement $Y \setminus \operatorname{int} V$ can be identified with $X \times I$ where $\partial Y = X = X \times 0$ and $\partial V = X \times 1$ $(I = [0, 1])$. Then the restriction map

$$g| : X \times I \to X \times I$$

is a pseudo-isotopy between $g|_{X\times 0} = f$ and $g|_{X\times 1} =$ identity (use 2) above). Thus the homeomorphism $f : X \to X$ is pseudo-isotopic to the identity as claimed.

∎

Corollary 10. *If* $X = \#_p \left(S^1 \times S^n\right)$, $n \geq 3$, $p \geq 1$, *then the natural map*

$$\mathcal{D}(X) \to \mathcal{E}(X)$$

is an isomorphism.

Proof. By Proposition 9 the map of the statement is injective. It is also surjective because each homotopy self-equivalence of X is homotopic to a PL homeomorphism by Theorem 8.

■

Theorem 11. *If* $X = \#_p\left(S^1 \times S^n\right)$, $n \geq 3$, $p \geq 1$, *then we have the following exact sequence*

$$0 \longrightarrow \operatorname{Ker}\theta_0 \simeq \bigoplus_p \boldsymbol{Z}_2 \longrightarrow \mathcal{D}_0(X) \simeq \mathcal{E}_0(X) \xrightarrow{\theta_0} \operatorname{Out}(\Pi_1) \longrightarrow 0,$$

i.e. any two orientation-preserving diffeomorphisms $f, g : X \to X$ *with*

$$f_* = g_* : \Pi_1 \to \Pi_1$$

are pseudo-isotopic provided certain obstructions

$$\alpha_i \in \Pi_1\left(\mathrm{SO}(n+1)\right) \simeq \boldsymbol{Z}_2$$

vanish, $1 \leq i \leq p$.

In order to prove Theorem 11 we need the following lemma.

Lemma 12. *Let* $f, g : X \to X$ *be two degree one maps.*

If $f_* = g_* : \Pi_1 \to \Pi_1$, *then* $f_* = g_* : \Pi_q \to \Pi_q$ *for all* $q \leq n$.

Proof. We observe that $\Pi_i(X) = 0$ for $1 < i < n$, hence $f_* = g_* : \Pi_q \to \Pi_q$ for all $q < n$. By [12], p. 341, the Poincarè duality and the relation $\deg(f) = \deg(g) = 1$, we have the following commutative diagrams

$$\begin{array}{ccccccc}
H_n(\widetilde{X};Z) & = & H_n(\widetilde{X};Z) & \xrightarrow[PD]{\simeq} & H_c^1(\widetilde{X};Z) & = & H_c^1(\widetilde{X};Z) \\
\tilde{f}_* \downarrow & & \downarrow \tilde{g}_* & & \tilde{f}^* \downarrow & & \downarrow \tilde{g}^* \\
H_n(\widetilde{X};Z) & = & H_n(\widetilde{X};Z) & \xrightarrow[PD]{\simeq} & H_c^1(\widetilde{X};Z) & = & H_c^1(\widetilde{X};Z)
\end{array}$$

$$\begin{array}{ccccccc}
H_c^1(\widetilde{X};Z) & = & H_c^1(\widetilde{X};Z) & \xrightarrow{\simeq} & H^1(\Pi_1;Z[\Pi_1]) & = & H^1(\Pi_1;Z[\Pi_1]) \\
\tilde{f}^* \downarrow & & \downarrow \tilde{g}^* & & f_*^* \downarrow & & \downarrow g_*^* \\
H_c^1(\widetilde{X};Z) & = & H_c^1(\widetilde{X};Z) & \xrightarrow{\simeq} & H^1(\Pi_1;Z[\Pi_1]) & = & H^1(\Pi_1;Z[\Pi_1])
\end{array}$$

where $\tilde{f}, \tilde{g} : \tilde{X} \to \tilde{X}$ are the liftings of f, g respectively.

Since the hypothesis $f_* = g_* : \Pi_1 \to \Pi_1$ directly implies $f_*^* = g_*^*$, it follows that $\tilde{f}_* = \tilde{g}_* : H_n(\tilde{X}; \boldsymbol{Z}) \to H_n(\tilde{X}; \boldsymbol{Z})$. Then the Hurewicz isomorphism

$$H_n(\tilde{X}; \boldsymbol{Z}) \simeq \Pi_n(\tilde{X}) \simeq \Pi_n(X)$$

implies that $f_* = g_* : \Pi_n \to \Pi_n$ as required.

∎

Proof of Theorem 11. Here we prove that the sequence

$$\oplus_p \boldsymbol{Z}_2 \to \mathcal{D}_0(X) \to \mathrm{Out}(\Pi_1) \to 0$$

is exact. The injectivity of the term $\oplus_p \boldsymbol{Z}_2$ into $\mathcal{D}_0(X)$ will follow from realizations of obstructions in section 4.

Suppose that $f : X \to X = \#_p(\boldsymbol{S}^1 \times \boldsymbol{S}^n)$, $n \geq 3$, is an orientation-preserving diffeomorphism such that $\theta_0(f) = 1$. We can choose a representative (also named f) in the class of f which preserves the base point of X and $f_* =$ identity on $\Pi_1(X)$. Lemma 12 implies that $f_* =$ identity on $\Pi_q(X)$ for all $q \leq n$. By Proposition 9 it is enough to show that f is homotopic (and hence pseudo-isotopic) to the identity $\mathrm{Id}_X : X \to X$.

We attempt to build up a homotopy $F : X \times I \to X$ between f and Id_X in steps, using a filtration of X by subcomplexes.

Consider the handle presentation

$$X = D^{n+1} \cup_\chi \bigcup_{i=1}^{p} \left(D_i^1 \times D_i^n\right) \cup_\psi \bigcup_{j=1}^{p} \left(D_j^n \times D_j^1\right) \cup B^{n+1}$$

where D, B are $(n+1)$-cells and χ, ψ are embeddings

$$\chi : \bigcup_{i=1}^{p} \left(\partial D_i^1\right) \times D_i^n \to \partial D^{n+1} = \boldsymbol{S}^n$$

$$\psi : \bigcup_{j=1}^{p} \left(\partial D_j^n\right) \times D_j^1 \to \partial \left(D^{n+1} \cup_\chi \bigcup_{i=1}^{p} \left(D_i^1 \times D_i^n\right)\right).$$

Our filtration starts with D^{n+1}, then we successively add $D_i^1 \times 0$, $D_i^1 \times D_i^n$, $D_j^n \times 0$, $D_j^n \times D_j^1$ and finally B^{n+1}.

Now we regard f as a diffeomorphism of $X \times 1$ and seek to extend f on $X \times 1$ and the identity Id on $X \times 0$ to a map $F : X \times I \to X$, where $I = [0, 1]$.

Step 1. By the disc theorem $f|_{D^{n+1}}$ and $\mathrm{Id}|_{D^{n+1}}$ are homotopic. Thus we choose a homotopy and define it

$$F|_{D^{n+1}\times I} : D^{n+1} \times I \to D^{n+1} \subset X.$$

Step 2. We next define F on $D_i^1 \times I$. Now $F|_{\partial(D_i^1\times I)}$ is already given: on $\partial D_i^1 \times I \subset D^{n+1} \times I$ by Step 1, on $D_i^1 \times 0$ by the identity and on $D_i^1 \times 1$ by f.
Because $f_* =$ identity on $\Pi_1(X)$, we can extend to some map

$$D_i^1 \times I \to X.$$

Indeed, let S_i^1 be the i-th S^1-factor of $X = \#_p \left(S^1 \times S^n\right)$, $n \geq 3$; the condition $f_* =$ identity on Π_1 implies that the 1-sphere $f(S_i^1)$ is homotopic to S_i^1 (and also isotopic as $\dim X \geq 4$).

Step 3. We now extend F to $D_i^1 \times D_i^n \times I$, i.e. to a tubular neighborhood of $D_i^1 \times I$ in $X \times I$. By the tubular neighborhood theorem it suffices to find a trivialisation of the normal bundle with the desired properties. As in step 2 these turn out to be that a trivialisation is already given on the boundary $\partial\left(D_i^1 \times I\right)$. The obstruction to extending this over $D_i^1 \times I$ (since this is contractible, the bundle certainly is trivial) is an element (see [17])

$$\alpha_i \in \Pi_1\left(\mathrm{SO}(n+1)\right) \simeq \mathbb{Z}_2$$

(see [1], [8] for the stable homotopy of the orthogonal group, $n \geq 3$).

If $\alpha_i = 0$, then the extension of the framing and hence of F is possible.

Step 4. We now assume that steps 1,2,3 have been successfully performed, i.e. F has been already defined on

$$\left(D^{n+1} \cup_X \bigcup_{i=1}^{p} \left(D_i^1 \times D_i^n\right)\right) \times I.$$

We next extend F on $D_j^n \times I$. Now $F|_{\partial(D_j^n\times I)}$ is already given:
on $\partial D_j^n \times I \subset \partial\left(D^{n+1} \cup_X \bigcup_{i=1}^{p} \left(D_i^1 \times D_i^n\right)\right) \times I$ by step 2, on $D_j^n \times 0$ by the identity and on $D_j^n \times 1$ by f.

Because $f_* =$ identity on $\Pi_n(X)$ (see Lemma 12) we can extend F to some map

$$D_j^n \times I \to X.$$

Indeed, let S_j^n $(j = 1, 2, \ldots, p)$ be the j-th S^n-factor of X. The condition $f_* =$ identity on $\Pi_n(X)$ implies that the n-sphere $f\left(S_j^n\right)$ is homotopic to S_j^n.

Step 5. We have now to extend F to $D_j^n \times D_j^1 \times I$, i.e. to a tubular neighborhood of $D_j^n \times I$ in $X \times I$. As remarked in step 3, the obstruction to extending a trivialisation given on the boundary $\partial\left(D_j^n \times I\right)$ to $D_j^n \times I$ is an element of $\Pi_n(\mathrm{SO}(2)) \simeq \Pi_n(S^1) \simeq 0$ for $n \geq 3$. Thus the extension of the framing and hence of F is possible on the whole of $(X \setminus \mathrm{int}(B^{n+1})) \times I$. Then we complete the extension of F to $X \times I$ by using the Alexander theorem.

Finally we prove that the homomorphism θ_0 is surjective. Indeed, for any $\xi \in \mathrm{Out}(\Pi_1)$ there exists $f \in \mathrm{Aut}(X)$ such that $f_* = \xi$ (see [12]). If $\deg(f) = 1$, then $[f] \in \mathcal{D}_0(X)$ and $\theta_0[f] = \xi$. Otherwise we compose f with the homeomorphism

$$r' = \#_p\left(\mathrm{Id}_{S^1} \times r\right) : X \to X$$

where $r : S^n \to S^n$ is the reflection on the 1-st coordinate. Then $[f \circ r'] \in \mathcal{D}_0(X)$ and $\theta_0[f \circ r'] = \xi$. Thus the proof is completed.

∎

In section 4 we will show that any obstructions can be realized.

3 Alternative proofs

We can give an alternative proof of Theorem 11 by applying the classical obstruction theory (compare for example with [6]).

In order to do this, we need some algebraic lemmas which are interesting by itself.

Lemma 13.

1) Let $\Lambda = \mathbf{Z}[\Pi_1]$ *be the group ring of* $\Pi_1(X)$. *Let* $e_1, e_2, \ldots, e_p \in \Pi_1(X)$ *be canonical generators and let*

$$\sigma = (e_1 - 1, e_2 - 1, \ldots, e_p - 1) \in \bigoplus_p \Lambda.$$

Then the Λ*-module* $\Pi_n(X)$ *is* Λ*-isomorphic to* $(\bigoplus_p \Lambda)/\sigma\Lambda$.

2) The Λ*-module* $\Pi_{n+1}(X)$ *is* Λ*-isomorphic to* $(\bigoplus_{p-1} \frac{\Lambda}{2\Lambda}) \oplus (\bigoplus_p \mathbf{Z}_2)$, *where* Λ *acts on* $\mathbf{Z}_2$ *via the map naturally induced by the augmentation* $\epsilon : \Lambda \to \mathbf{Z}$.

Proof.

1) Let $X^{(q)}$ be the q-skeleton of the standard cellular decomposition

$$e^0 \cup pe^1 \cup pe^n \cup e^{n+1}$$

of X. Since $X^{(q)} = X^{(1)}$ for $1 \leq q < n$, we have

$$\Pi_n(X) \simeq \Pi_n(\tilde{X}) \simeq H_n(\tilde{X}; \mathbf{Z}) \simeq H_n(X; \Lambda)$$

and $H_n(X;\Lambda) \simeq H^1(X;\Lambda)$ by Poincarè duality. Here $\tilde{X}$ denotes the universal covering space of X.

To calculate $H^1(X;\Lambda)$ we consider the exact sequence

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H_1\left(\tilde{X}^{(1)}, \tilde{X}^{(0)}\right) & \longrightarrow & H_0\left(\tilde{X}^{(0)}\right) & \longrightarrow & H_0\left(\tilde{X}^{(1)}\right) & \longrightarrow & 0 \\
 & & \text{iso} \downarrow & & \text{iso} \downarrow & & \text{iso} \downarrow & & \\
0 & \longrightarrow & I(\Lambda) & \xrightarrow{i} & \Lambda & \xrightarrow{\epsilon} & \mathbf{Z} & \longrightarrow & 0,
\end{array}$$

which gives the following augmented Λ-chain complex

$$0 \longrightarrow \mathrm{Hom}_\Lambda(\mathbf{Z}, \Lambda) \xrightarrow{\epsilon^\#} \mathrm{Hom}_\Lambda(\Lambda, \Lambda) \xrightarrow{i^\#} \mathrm{Hom}_\Lambda(I(\Lambda), \Lambda) \longrightarrow 0,$$

hence

$$H^1(X;\Lambda) \simeq \operatorname{coker} i^\# \simeq \frac{\mathrm{Hom}_\Lambda(I(\Lambda), \Lambda)}{\mathrm{Im}\, i^\#}.$$

As Λ-module, $I(\Lambda)$ is isomorphic to

$$\Lambda(e_1 - 1) \oplus \Lambda(e_2 - 1) \oplus \ldots \oplus \Lambda(e_p - 1),$$

where

$$e_1, e_2, \ldots, e_p \in \Pi_1(X) \simeq \underset{p}{*} \boldsymbol{Z}$$

are canonical generators. If $\varphi \in \mathrm{Hom}_\Lambda(\Lambda, \Lambda)$, then $i^{\#}(\varphi)$ corresponds to

$$\sigma\varphi(1) \in \underset{p}{\oplus} \Lambda \simeq \mathrm{Hom}_\Lambda(I(\Lambda), \Lambda),$$

proving statement 1) of the lemma.

2) Let X^* be the CW-complex obtained from $X = \#_p \left(S^1 \times S^n\right)$ by attaching $p-1$ $(n+1)$-cells D^{n+1} along the n-spheres where the connected sum is taken. Observe that X^* is homotopy equivalent to the wedge $\underset{p}{\vee} \left(S^1 \times S^n\right)$.

Furthermore one can easily verify the following isomorphisms:

$$\Pi_{n+1}(X^*) \simeq \Pi_{n+2}(X^*) \simeq \underset{p}{\oplus} \boldsymbol{Z}_2$$

$$\Pi_{n+1}(X^*, X) \simeq \underset{p-1}{\oplus} \Lambda \qquad \Pi_{n+2}(X^*, X) \simeq \underset{p-1}{\oplus} \frac{\Lambda}{2\Lambda}.$$

Thus the homotopy exact sequence of the pair (X^*, X) yields

$$\Pi_{n+2}(X) \xrightarrow{j_*} \Pi_{n+2}(X^*) \longrightarrow \Pi_{n+2}(X^*, X)$$
$$\longrightarrow \Pi_{n+1}(X) \longrightarrow \Pi_{n+1}(X^*) \longrightarrow 0.$$

Since j_* is an epimorphism, we obtain the result.

■

Given a Λ-module L, we denote by $H^*(X; L)$ the cohomology of the complex $\mathrm{Hom}_\Lambda(C_*(\tilde{X}), L)$, where $C_*(\tilde{X}) = H_*(\tilde{X}^{(*)}, \tilde{X}^{(*-1)})$.

Lemma 14.

1) $H^n(X; \Pi_n(X)) \simeq \boldsymbol{Z}$

2) $H^{n+1}(X; \Pi_{n+1}(X)) \simeq \underset{p}{\oplus} \boldsymbol{Z}_2.$

Proof.

1) By Poincarè duality, we have $H^n(X;\Pi_n(X)) \simeq H_1(X;\Pi_n(X))$ (see [18]).

Using $\Pi_n(X) \simeq (\bigoplus_p \Lambda)/\sigma\Lambda$, one obtains the following exact sequence

$$\begin{aligned} H_1(X;\Lambda) \longrightarrow H_1(X;\bigoplus_p \Lambda) \longrightarrow H_1(X;(\bigoplus_p \Lambda)/\sigma\Lambda) \\ \longrightarrow \boldsymbol{Z} \otimes_\Lambda \Lambda \longrightarrow \boldsymbol{Z} \otimes_\Lambda (\bigoplus_p \Lambda). \end{aligned}$$

Now $H_1(X;\Lambda) \simeq H_1(X;\bigoplus_p \Lambda) \simeq 0$ and $\boldsymbol{Z} \otimes_\Lambda \Lambda \to \boldsymbol{Z} \otimes_\Lambda (\bigoplus_p \Lambda)$ is the null homomorphism because σ goes to zero. Hence we obtain

$$H^n(X;\Pi_n(X)) \simeq \boldsymbol{Z} \otimes_\Lambda \Lambda \simeq \boldsymbol{Z}$$

as claimed (use also [15], Theorem 1.12).

2) We have

$$H^{n+1}(X;\Pi_{n+1}(X)) \underset{\mathrm{PD}}{\simeq} H_0(X;\Pi_{n+1}(X)) \simeq \Pi_{n+1}(X)_{\Pi_1(X)}$$

where $\Pi_{n+1}(X)_{\Pi_1(X)}$ is the maximal quotient module of $\Pi_{n+1}(X)$ (see [15], p. 266), i.e.

$$\Pi_{n+1}(X)_{\Pi_1(X)} = \frac{\Pi_{n+1}(X)}{\{\lambda x : \lambda \in \Lambda \wedge x \in \Pi_{n+1}(X)\}}.$$

Because this quotient module is Λ-trivial (see [15]), Lemma 13 implies that

$$H^{n+1}(X;\Pi_{n+1}(X)) \simeq \bigoplus_p \boldsymbol{Z}_2.$$

Thus the proof is completed.

∎

Theorem 11: second proof. Let $f : X \to X = \#_p\left(S^1 \times S^n\right)$, $n \geq 3$, $p \geq 1$, be a homotopy self-equivalence of degree one such that $\theta_0(f) = 1$. As before, we can assume that f preserves the base point of X and that $f_* = (\mathrm{Id}_X)_*$ on $\Pi_q(X)$ for all $q \leq n$ (see Lemma 12). We have to study under that conditions f is homotopic to the identity Id_X. We attempt to build up a homotopy $h : X \times I \to X$ between f and Id_X in steps using a filtration of X by subcomplexes.

Let $X^{(q)}$ be the q-skeleton of the standard cellular decomposition

$$X = e^0 \cup pe^1 \cup pe^n \cup e^{n+1}.$$

Because $f_* = (\mathrm{Id}_X)_*$ on Π_1, there is a homotopy

$$h : X^{(2)} \times I \to X$$

between $f|_{X^{(2)}}$ and $\mathrm{Id}_X|_{X^{(2)}}$.

The equalities $X^{(q)} = X^{(1)}$ for $1 \leq q < n$ imply that

$$H^q(X; \Pi_q(X)) \simeq 0$$

for all $1 \leq q < n$. Thus the first obstruction lies in

$$H^n(X; \Pi_n(X)) \simeq \boldsymbol{Z}$$

(see Lemma 14). Let $h : X^{(n-1)} \times I \to X$ be a homotopy between $f|_{X^{(n-1)}}$ and $\mathrm{Id}_X|_{X^{(n-1)}}$. The obstruction to extend h to $X^{(n)}$ is the homotopy class of the map

$$f \cup h \cup \mathrm{Id}_X : X \times 0 \cup X^{(n-1)} \times I \cup X \times 1 \to X,$$

i.e. for any $i = 1, 2, \ldots, p$ we have

$$\Delta_i(f, h, \mathrm{Id}_X) = \left[f \cup h \cup \mathrm{Id}_X |_{e_i^n \times 0 \cup \partial e_i^n \times I \cup e_i^n \times 1} \right] \in \Pi_n(X).$$

In other words, the difference cochain is defined as follows:

$$\begin{aligned} d(f, h, \mathrm{Id}_X) : \ & C_n(X) \to \Pi_n(X) \\ & e_i^n \to \Delta_i(f, h, \mathrm{Id}_X) \end{aligned}$$

hence the obstruction is the cohomology class

$$[d(f, h, \mathrm{Id}_X)] \equiv [\sum_{i=1}^{p} \Delta_i] \in H^n(X; \Pi_n(X)) \simeq \boldsymbol{Z}.$$

Let $h' : X^{(n-1)} \times I \to X$ be a homotopy of $\mathrm{Id}|_{X^{(n-1)}}$ to $\mathrm{Id}|_{X^{(n-1)}}$. It is well-known that

$$\Delta(f, h, \mathrm{Id}_X) + \Delta(\mathrm{Id}_X, h', \mathrm{Id}_X) = \Delta(f, h + h', \mathrm{Id}_X)$$

where $h + h' : X^{(n-1)} \times I \to X$ is defined by

$$(h + h')(x,t) = \begin{cases} h(x,2t) & 0 \le t \le \frac{1}{2} \\ h'(x,2t-1) & \frac{1}{2} \le t \le 1. \end{cases}$$

Indeed, for each $i = 1,2,\ldots,p$, we take a small ball in the centre of the n-cell e_i^n and cut off it. Next we attach to its place a spheroid representing the value in $\Pi_n(X)$ of the cochain d at e_i^n. Thus we can always choose an h' such that

$$d(\mathrm{Id}_X, h', \mathrm{Id}_X) = -d(f, h, \mathrm{Id}_X),$$

i.e. $h + h'$ extends to a homotopy $X^{(n)} \times I \to X$ between $f|_{X^{(n)}}$ and $\mathrm{Id}_X|_{X^{(n)}}$. Now the only obstructions lie in

$$H^{n+1}(X; \Pi_{n+1}(X)) \simeq \bigoplus_p \mathbf{Z}_2.$$

This proves Theorem 11.

∎

4 Realizing obstructions

Now we are going to prove Theorem 3.

Let $\{e_i\}$, $i = 1,2,\ldots,p$, be a free basis of $\Pi_1(X) \simeq \ast_p \mathbf{Z}$, where $X = \#_p \left(S^1 \times S^n\right)$, $p \ge 1$, $n \ge 3$. Obviously e_i is the homotopy class of the i-th S^1-factor S_i^1 of X. As proved in [10] and [12], $\mathrm{Aut}(\Pi_1)$ is generated by sliding 1-handles, twisting 1-handles and permuting 1-handles. More precisely, for $i = 2,3,\ldots,p$ ($p > 1$) define $\phi_i \in \mathrm{Aut}(\Pi_1)$ by setting $\phi_i(e_1) = e_i$, $\phi_i(e_i) = e_1$ and $\phi_i(e_j) = e_j$ for each $j \ne i$, $j \ne 1$. Permuting the 1-handles e_i and e_j corresponds to the automorphism $\phi_i \circ \phi_j \circ \phi_i^{-1}$. It follows that $\phi_i^2 = 1$ and by [10], [12] there exist diffeomorphisms $f_i : X \to X$ (*permuting 1-handles*) such that $f_{i*} = \phi_i$. Then define $\sigma \in \mathrm{Aut}(\Pi_1)$ by setting $\sigma(e_1) = e_1^{-1}$ and $\sigma(e_i) = e_i$ for $i \ne 1$. Twisting the 1-handle e_i corresponds to the automorphism $\phi_i \circ \sigma \circ \phi_i^{-1}$. Obviously $\sigma^2 = 1$. Furthermore there exist diffeomorphisms of X (*twisting 1-handles*) which realize σ and $\phi_i \circ \sigma \circ \phi_i^{-1}$ for $i \ge 2$. Finally we define

$\psi \in \mathrm{Aut}(\Pi_1)$, $p > 1$, by setting $\psi(e_1) = e_1 e_2$ and $\psi(e_i) = e_i$ for $i \geq 2$ (*sliding 1-handles*).

Let $\Sigma_i = \boldsymbol{S}^n_i$ be the i-th $\boldsymbol{S}^n$-factor of $X = \#_p \left(\boldsymbol{S}^1 \times \boldsymbol{S}^n\right)$, $p \geq 1$, $n \geq 3$. Following [10], we show that rotations of X parallel to Σ_i generate the obstruction subgroup

$$\mathrm{Ker}\,\theta_0 \simeq \bigoplus_p \Pi_1\left(\mathrm{SO}(n+1)\right) \simeq \bigoplus_p \boldsymbol{Z}_2.$$

Let

$$\alpha : (\boldsymbol{S}^1, 1) \to (\mathrm{SO}(n+1), \mathrm{id})$$

be a loop representing a homotopy class of $\Pi_1\left(\mathrm{SO}(n+1)\right) \simeq \boldsymbol{Z}_2$ $(n \geq 3)$.

Then α induces a diffeomorphism

$$h_\alpha : \boldsymbol{S}^n \times I \to \boldsymbol{S}^n \times I$$

defined by

$$h_\alpha(x,t) = (\alpha(t)x, t)$$

for all $x \in \boldsymbol{S}^n$ and $t \in I = [0,1]$. Obviously h_α is the identity on the boundary $\partial(\boldsymbol{S}^n \times I) = \boldsymbol{S}^n \times 0 \cup \boldsymbol{S}^n \times 1$.

Now let M^{n+1} be a closed oriented $(n+1)$-manifold and let Σ^n be an oriented n-sphere embedded in M. Suppose $\varphi : \boldsymbol{S}^n \times I \to M$ is an orientation-preserving embedding such that $\varphi(\boldsymbol{S}^n \times 0) = \Sigma$. Because $h_\alpha =$ identity on $\partial(\boldsymbol{S}^n \times I)$, one obtains a diffeomorphism

$$h^\Sigma_\alpha : M \to M$$

defined by

$$h^\Sigma_\alpha(x) = \begin{cases} x & \text{if} \quad x \in M \setminus \mathrm{Im}\,\varphi \\ \varphi \circ h \circ \varphi^{-1}(x) & \text{if} \quad x \in \mathrm{Im}\,\varphi. \end{cases}$$

We call the diffeomorphism h^Σ_α the *α-rotation of M parallel to Σ* (briefly, a *rotation*). Obviously the pseudo-isotopy class of h^Σ_α depends only on the homotopy (resp. isotopy) class of α (resp. Σ).

If $M^{n+1} = X = \#_p \left(\boldsymbol{S}^1 \times \boldsymbol{S}^n\right)$, $p \geq 1$, $n \geq 3$, then let $\Sigma_i = \boldsymbol{S}^n_i$ be the i-th $\boldsymbol{S}^n$-factor of X. We set

$$h_{i,\alpha} = h^{\Sigma_i}_\alpha$$

for $i = 1, 2, \ldots, p$ and $[\alpha] \in \Pi_1(\mathrm{SO}(n+1)) \simeq \mathbf{Z}_2$. One can choose $h_{i,\alpha}$ to be the identity on the union $\cup_{i=1}^p \Sigma_i$. Because $(h_{i,\alpha})_* =$ identity on $\Pi_q(X)$ for all $q \leq n$, we have that $h_{i,\alpha} \in \operatorname{Ker}\theta_0$, $i = 1, 2, \ldots, p$. Moreover $h_{i,\alpha} \circ h_{j,\beta} = h_{j,\beta} \circ h_{i,\alpha}$ $(i \neq j)$, each $h_{i,\alpha}$ commutes with the generators of $\mathrm{Aut}(\Pi_1)$ and $h_{i,\alpha}$ is pseudo-isotopic to the identity if and only if $[\alpha] = 0$. Thus we have shown that the rotations $h_i = h_{i,\alpha}$ of X parallel to the n-spheres Σ_i generate $\operatorname{Ker}\theta_0$ if $[\alpha]$ is the generator of $\Pi_1(\mathrm{SO}(n+1)) \simeq \mathbf{Z}_2$. In particular, this shows that the term $\bigoplus_p \mathbf{Z}_2$ injects into $\mathcal{D}_0(X)$.

More precisely, we can interpret our results in the following way (which is related to Lemma 5.4 of [10]):

Corollary 15. *Let* $X = \#_p\left(S^1 \times S^n\right)$, $p \geq 1$, $n \geq 3$, *and let* $f : X \to X$ *be an orientation-preserving diffeomorphism such that* $\theta_0(f) = 1$, *i.e.* $f_* =$ identity *on* Π_1. *Then there exist loops (obstructions)*

$$\alpha_i : (S^1, 1) \to (\mathrm{SO}(n+1), \mathrm{Id})$$

$(i = 1, 2, \ldots, p)$ *such that* f *is pseudo-isotopic to the product*

$$h_{1,\alpha_1} \circ h_{2,\alpha_2} \circ \ldots \circ h_{p,\alpha_p}.$$

Moreover, the pseudo-isotopy can be chosen keeping the union $\cup_{i=1}^p \Sigma_i$ *fixed.*

In other words, rotations $h_i = h_{i,\alpha}$ $(i = 1, 2, \ldots, p)$ *is a free basis of*

$$\operatorname{Ker}\theta_0 \simeq \bigoplus_p \Pi_1(\mathrm{SO}(n+1)) \simeq \bigoplus_p \mathbf{Z}_2$$

where $[\alpha]$ *is the generator of* $\Pi_1(\mathrm{SO}(n+1)) \simeq \mathbf{Z}_2$.

5 Concluding remarks

Following [12], let $\mathcal{C}(n, \lambda)$ denote the class of smooth $(n+1)$-manifolds of the form

$$N^{n+1} = H^0 \cup pH^\lambda \cup pH^{\lambda+1}$$

such that N is contractible, $n \geq 3$, $1 \leq \lambda \leq n-1$. The h-cobordism theorem of Smale implies that if $N \in \mathcal{C}(n, \lambda)$, then N is an $(n+1)$-disc provided that $n \geq 5$ and $1 \leq \lambda \leq n-3$ (see for example [16]). On the other hand $C(n, n-2)$ contains elements with non-simply connected boundary (see [12]).

Now one might ask the following question:

$$N \in C(n, n-1) \Longrightarrow N \underset{diff}{\simeq} D^{n+1}.$$

We can apply Corollary 6 to give a positive answer in the particular case

$$H^0 \cup pH^{n-1} \simeq \#_p \left(S^{n-1} \times D^2\right).$$

Indeed, we have the following result.

Proposition 16. *Let N^{n+1} be the manifold obtained by attaching p handles of index n to $\#_p \left(S^{n-1} \times D^2\right)$, $n \geq 3$. If $H_{n-1}(N; \mathbf{Z}) = 0$, then ∂N is diffeomorphic to the n-sphere S^n.*

Proof. We simply follow the proof of Lemma 5 [12], settled for $n = 3$. First of all, the hypothesis $H_{n-1}(N; \mathbf{Z}) = 0$ implies that N is contractible, i.e. $N \in C(n, n-1)$.

Let ψ_n^i be the attaching map of the i-th handle $H_i^n = D_i^n \times D_i^1$ of index n, $i = 1, 2, \ldots, p$. The same argument as in [12] shows that $\psi_n^i \left(\partial D_i^n \times \frac{1}{2}\right)$ are disjoined homologically independent $(n-1)$-spheres embedded in

$$\partial \left(\#_p (S^{n-1} \times D^2)\right) = \#_p (S^{n-1} \times S^1).$$

Let Σ_i^{n-1} be the i-th $(n-1)$-factor of $\#_p \left(S^{n-1} \times S^1\right)$. Cutting $\#_p \left(S^{n-1} \times S^1\right)$ along the $(n-1)$-spheres $\psi_n^i \left(\partial D_i^n \times \frac{1}{2}\right)$ (resp. Σ_i^{n-1}) yields a punctured n-disc P^n (resp. Q^n), where

$$P^n \simeq Q^n \simeq D^n \backslash \{2p - 1\, \text{open}\, n - \text{cells}\}.$$

A diffeomorphism $P^n \to Q^n$ which preserves the boundary components induces a diffeomorphism between the pairs

$$(\#_p \left(S^{n-1} \times S^1\right), \bigcup_i \psi_n^i (\partial D_i^n \times \frac{1}{2}))$$

and

$$(\#_p\left(S^{n-1} \times S^1\right), \bigcup_i \Sigma_i^{n-1}).$$

This implies the statement.

■

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