A short intervals result for linear equations in two prime variables.

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Abstract

Given $A$ and $B$ integers relatively prime, we prove that almost all integers $n$ in an interval of the form $[N, N+H]$, where $N^{1/3+\epsilon} \leq H \leq N$, can be written as a sum $Ap_1 + Bp_2 = n$, with $p_1$ and $p_2$ primes and $\epsilon$ an arbitrary positive constant. This generalizes the results of [PP] established in the classical case $A = B = 1$ (Goldbach's problem).

1 Introduction

Let us consider the linear equation $Ap_1 + Bp_2 = n$, where $p_1$ and $p_2$ are prime numbers and $A$, $B$, $n$ are positive integers satisfying the following conditions:

1) $A$, $B$ are relatively prime, $(A, B) = 1$,

2) $n \in \mathcal{A} = \{n : (AB, n) = 1, ABn \equiv 0 \pmod{2}\}$.

In the classical case $A = B = 1$ (Goldbach's problem) Perelli and Pintz [PP] proved that the above equation has solutions for all even integers $2n$ in an interval of the form $[N, N+H]$ with $O(HL^{-E})$ exceptions

AMS Classification: 11P32.

Partially supported by 40% MURST Grant.

and

\[ R(2n) = 2n\mathcal{S}(2n) + O(NL^{-C}), \]

where

\[
R(2n) = \sum_{h+k=2n} \Lambda(h)\Lambda(k), \quad \mathcal{S}(2n) = 2\prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right) \prod_{p|n} \left(1 - \frac{1}{p-2}\right).
\]

\(\Lambda\) is von Mangoldt's function, \(L = \log N\), \(N^{1/3+\varepsilon} \leq H \leq N\) and \(E, C > 0, 0 < \varepsilon < 2/3\) are arbitrary constants.

In this paper we want to establish the same results in the general case for \(A\) and \(\varepsilon\). Let us denote

\[
R(n) = R(n, A, B) = \sum_{Ah+Bk=n} \Lambda(h)\Lambda(k),
\]

\[
M(n) = M(n, A, B) = \sum_{Ah+Bk=n} 1,
\]

\[
\mathcal{S}(n) = \mathcal{S}_{\varepsilon,A}(n) = \prod_{p|ABn} \left(1 + \frac{1}{p-1}\right) \prod_{p|ABn} \left(1 - \frac{1}{(p-1)^2}\right).
\]

**Theorem 1.** Let \(0 < \varepsilon < 2/3, C > 0\) be arbitrary constants and \(N^{1/3+\varepsilon} \leq H \leq N\). Then

\[
\sum_{n < n \leq N+H} \left| R(n) - M(n)\mathcal{S}(n) \right|^2 \ll_{\varepsilon,A,C} HN^2L^{-C}.
\]

Clearly Theorem 1 implies

**Corollary.** Let \(E, C > 0\) be arbitrary constants and \(N^{1/3+\varepsilon} \leq H \leq N\). Then, for all \(n \in [N, N+H] \cap A\) with at most \(O(HL^{-E})\) exceptions, the equation \(Ap_1 + Bp_2 = n\) is solvable and we have

\[ R(n) = M(n)\mathcal{S}(n) + O_{\varepsilon,A,C}(NL^{-C}). \]

**Theorem 2.** Let \(0 < \varepsilon < 5/6, E > 0\) be arbitrary constants. Then, for all \(n \in [N, N+H] \cap A\) with at most \(O_{\varepsilon,A,C}(HL^{-E})\) exceptions, the equation \(Ap_1 + Bp_2 = n\) is solvable, provided \(N^{7/96+\varepsilon} \leq H \leq N\).
Remarks.

I. The condition \((A, B) = 1\) is natural. In fact, if \((A, B) = d > 1\), then the equation \(Ap_1 + Bp_2 = n\) has no solutions when \(d|n\). If \(d|n\), then the equation is equivalent to \(A'p_1 + B'p_2 = n'\), where \(A' = A/d, B' = B/d, n' = n/d\) with \((A', B') = 1\). If \(n \not\in A\), then our equation has at most one solution, and our method is not able to detect it.

II. It is easy to see [A; ch.2, ex.9] that for fixed \(A\) and \(B\), \((A, \phi) = 1\), we have

\[
|\{N, N + H\} \cap A| \sim \frac{\varphi(AB)}{AB} H,
\]

where \(\varphi\) is Euler's function. Hence the exceptional set in the above results is of order smaller than \(|\{N, N + H\} \cap A|\).

III. The number of non-negative integer solutions of the linear equation \(Ax + By = n\), with \((A, B) = 1\), is given by

\[
M(n) = \left[ \frac{n}{AB} \right] \text{ or } \left[ \frac{n}{AB} \right] + 1.
\]

This result can be proved by appealing to the following theorem (see [NZM; Theorem 5.1]) and recalling that \([\alpha - [\beta]] = [\alpha - \beta] \text{ or } [\alpha - \beta] + 1\) (where \([\alpha]\) denotes the integer part of \(\alpha\)).

Theorem 3. Let \((x_0, y_0)\) be an integer solution of the linear equation \(Ax + By = n\), with \((A, B) = 1\). Then the solutions of \(Ax + By = n\) are given by:

\[
x = x_0 + Bt, \quad y = y_0 - At,
\]

where \(t\) is an integer.

IV. Following the proofs of Theorem 1 and 2 below and computing the implicit constant in \(\ll_{E,A,B,C}\), it is easy to remark that the aforementioned results holds uniformly for \(A, B \ll L^{C_1}\), for a suitable constant \(C_1 = C_1(C) > 0\).
2 Notation

$C$ - an arbitrary positive constant,
$\varepsilon$ - an arbitrarily small positive constant,
$\delta = 1/3 + \varepsilon$ or $7/36 + \varepsilon$,
$N, H$ - positive integers such that $N^\delta \leq H \leq N$, $N > N_0 = N_0(A, B, C, \varepsilon)$,
$D = \frac{3}{4}(2C + 25)$, $Q = \sqrt{H}$.

\[
I_{q,a} = \left\{ \frac{a}{q} + \eta, \eta \in \xi_{q,a} \right\}, \text{ where } \xi_{q,a} \subset \left( -\frac{1}{qQ}, \frac{1}{qQ} \right),
\]

\[
I'_{q,a} = \left\{ \frac{a}{q} + \eta, \eta \in \xi'_q \right\}, \text{ where } \xi'_q = \left( -\frac{LAD}{qY}, \frac{LAD}{qY} \right),
\]

\[
\mathcal{M} = \bigcup_{q \leq LD} \bigcup_{a = 1}^{q} I'_{q,a} \text{ major arcs, } m = \left[ \frac{1}{Q}, 1 + \frac{1}{Q} \right] \setminus \mathcal{M} \text{ minor arcs,}
\]

\[
I''_{q,a} = \{ \alpha \in [0, 1] | \exists \beta \in I'_{q,a} : B\alpha \equiv \beta \pmod 1 \} = B^{-1} \bigcup_{r=0}^{B-1} (r + I'_{q,a}),
\]

\[
\mathcal{M}_a = \bigcup_{q \leq LD} \bigcup_{a = 1}^{q} I''_{q,a}, \quad m_a = [0, 1] \setminus \mathcal{M}_a.
\]

\[
R^*(n) = R^*(N, Y, n, A, B) = \sum_{\substack{Ah+Bk = n \\ N - Y < Ah \leq N \\ Bk \leq Y}} \Lambda(h)\Lambda(k),
\]

\[
M^*(n) = M^*(N, Y, n, A, B) = \sum_{\substack{Ah+Bk = n \\ N - Y < Ah \leq N \\ Bk \leq Y}} 1.
\]

\[
e(\alpha) = e^{2\pi i \alpha}, \quad e_q(a) = e(a/q),
\]

\[
S_A(\alpha) = \sum_{N - Y < Ah \leq N} \Lambda(h)e(Ah\alpha), \quad S_B(\alpha) = \sum_{Bk \leq Y} \Lambda(k)e(Bk\alpha),
\]

\[
S(\beta) = \sum_{Bk \leq Y} \Lambda(k)e(k\beta), \quad T(\eta) = \sum_{Bk \leq Y} e(k\eta),
\]

\[
R(\eta, q, a) = S\left(\frac{a}{q} + \eta\right) - \frac{\mu(q)}{\varphi(q)} T(\eta),
\]
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\[ W(x, \eta) = \sum_{Bk \leq Y} \Lambda(k) \chi(k) \varepsilon(k\eta) - \delta X T(\eta), \]

with \( \delta_X = \begin{cases} 1 & \text{if } \chi = \chi_0, \\ 0 & \text{if } \chi \neq \chi_0. \end{cases} \)

\[ \sum_{d \mid q, \delta = 1} \sum_{a=1}^q, \| \beta \| = \text{distance of } \beta \text{ from the nearest integer}, \quad d(q) = \sum_{d \mid q}. \]

\[ c_q(m) = \sum_{a=1}^q e \left( \frac{ma}{q} \right), \quad \text{Ramanujan's sum} \]

\[ \gamma(\chi) = \sum_{a=1}^q \chi(a) e \left( \frac{a}{q} \right), \quad \text{Gauss' sum} \]

\[ \psi(x, \chi) = \sum_{n \leq x} \Lambda(n) \chi(n). \]

The constants in the \( \ll \) and \( O \)-symbols might depend on \( \varepsilon, A, B, C \), even in an ineffective way.

3 Lemmas

Lemma 1. Let \( 0 < \varepsilon < 5/12 \), \( D, G > 0 \) be arbitrary constants, and \( N \) be a sufficiently large integer. Suppose that \( N^{7/12+\varepsilon} \leq Y \leq N \) and \( q \leq L^D \). We have

\[ \sum_{Y < n \leq N} \Lambda(n) = Y + O_{\varepsilon, D}(Y L^{-G}), \]

for \( (a, q) = 1 \).

Proof. We have (see [PPS])

\[ \psi(N, \chi) - \psi(N - Y, \chi) = \delta_X Y + O_{\varepsilon, \varphi}(Y L^{-G}), \quad \text{for every } \chi \text{ (mod } q), \]

with \( G > 0, N^{7/12+\varepsilon} \leq Y \leq N, q \leq L^D \).
Then the Lemma follows from the property
\[
\sum_{\substack{N < h < N + h \equiv 0 \mod q}} \Lambda(h) = \frac{1}{\varphi(q)} \sum_{\chi} \chi(a)(\psi(N, \chi) - \psi(N - Y, \chi)).
\]

Lemma 2. If \((A, B) = 1\) and \(n \in A\), we have
\[
\prod_{p\mid An} \left(1 + \frac{1}{p-1}\right) \prod_{p\mid ABn} \left(1 - \frac{1}{(p-1)^2}\right) = \sum_{q=1}^{\infty} \frac{\mu(q)}{\varphi(q)^2} c_q(n) f_{A,B}(q),
\]
where \(f_{A,B}(q) = \mu(q(B))\mu(q, A)\varphi(q, A)\).

Proof. As in [V; p.34], using the well-known property \(c_q(n) = \frac{\mu(q(n))\varphi(q)}{\varphi(q(n))}\), we obtain
\[
\left\ll\sum_{\substack{X < q \leq Y \leq X + 1 \in A}} \frac{\mu(q)}{\varphi(q)^2} q(n)f_{A,B}(q)\right\ll = \sum_{\substack{t \leq X \in A}} \mu(tu)^2 \sum_{\substack{X < q \leq Y \leq X + 1 \in A}} \frac{\mu(q)}{\varphi(q)^2}
\ll \sum_{d\mid An} \frac{\mu(d)}{\varphi(d)} \min(\frac{d}{X}, 1).
\]
Hence \(\sum_{q=1}^{\infty} \frac{\mu(q)}{\varphi(q)^2} c_q(n)f_{A,B}(q)\) converges. Furthermore, it is easy to see that
\[
\frac{\mu(p^s)}{\varphi(p^s)^2} c_{p^s}(n)f_{A,B}(p^s) = \begin{cases} 
1, & \text{if } s = 0, \\
\frac{1}{\varphi(p)^2}, & \text{if } s = 1 \text{ and } p\mid An, \\
\frac{-1}{\varphi(p)^2}, & \text{if } s = 1 \text{ and } p \nmid ABn, \\
0, & \text{if } s = 1 \text{ and } p\mid B \text{ or } s > 1.
\end{cases}
\]
Then, the Lemma is proved.

Lemma 3. Let \(X\) and \(Y\) be natural numbers such that \(X < Y\). If \((A, B) = 1\) then for every real \(\xi \neq 0\) we have that
\[
\sum_{\substack{X < t \leq Y \leq X + 1 \in A}} e(t\xi) \ll \min\left(\frac{Y - X}{\delta AB}, 1, \frac{1}{\delta AB \xi}\right).
\]
where \( \hat{\delta} = \frac{3 - (-1)^{AB}}{2} \).

Proof. This is well-known for \( A = B = 1 \) (see [Vi]). Let \( h_1, h_2, \ldots, h_{\varphi(\hat{\delta}AB)} \) be a reduced residue system mod \( \hat{\delta}AB \).

Define \( h_i' = \min \{ X < t \leq Y : t \equiv h_i \pmod{\hat{\delta}AB} \} \), for every \( i = 1, 2, \ldots, \varphi(\hat{\delta}AB) \). Then we write

\[
\sum_{X < t < Y \atop t \in A} e(t \xi) = \sum_{i=1}^{\varphi(\hat{\delta}AB)} e(h'_i \xi) \sum_{0 \leq s < \frac{Y - X}{\hat{\delta}AB}} e(s \hat{\delta}AB \xi) \ll \\
\ll \varphi(\hat{\delta}AB) \min \left( \frac{Y - X}{\hat{\delta}AB}, \frac{1}{\| \hat{\delta}AB \xi \|} \right).
\]

Thus, the Lemma is proved.

Lemma 4. Let us denote

\[
I_{q,a}^u = \begin{cases} 
I_{q,a} \setminus I_{q,a}' & \text{if } q \leq L^D, \\
I_{q,a} & \text{if } q > L^D.
\end{cases}
\]

We have that

\[
\max_{(a,q) = 1} \int_{I_{q,a}^u} |S(\beta)|^2 d\beta \ll YL^{2C-3}.
\]

Proof. See [PP; §5].

4 Proof of Theorems 1 and 2

We note that \( M^*(n) = \frac{N + Y - 2n + 1}{AB} + O(1) \). Moreover, in the case \( Y = N \), for \( N < n \leq N + H \) we have that \( R^*(n) = R(n) + O_{A,B}(HL) \), \( M^*(n) = M(n) + O_{A,B}(HL) \). Hence, from \( E(n) \ll L \) we get that

\[
\sum_{N < n \leq N + H \atop n \in A} |R(n) - M(n)E(n)|^2 \ll \\
\sum_{N < n \leq N + H \atop n \in A} |R^*(n) - M^*(n)E(n)|^2 + H^3L^2 =
\]
Moreover, writing

\[ \sum_{m} = \sum_{N < n \leq N + H} \left| \int_{m_n} S_s(\alpha) S_p(\alpha) e(-n\alpha) d\alpha \right|^2, \]

we have

\[ \sum_{N < n \leq N + H} \left| R^*(n) - \frac{Y}{AB} \mathcal{G}(n) \right|^2 \ll \sum_{m} + \sum_{m_n}. \]

In order to prove Theorems 1 and 2 it suffices to choose \( H = Y^{1/3+\epsilon} \), where \( Y = N \) in Theorem 1 and \( Y = N^{7/12+\epsilon} \) in Theorem 2 and show that

1. \( \sum_{m_n} \ll H Y^2 L^{-C}, \)
2. \( \sum_{m} \ll H Y^2 L^{-C}. \)

Proof of (1). In this section we show that (1) holds for \( N^{7/12+\epsilon} \leq Y \leq N \). For any \( \alpha \in I_{q,a}^t \), we have that

\[ S_s(\alpha) = S(\beta) = \sum_{Bk \leq Y} \Lambda(n) e(k\beta), \]

where \( \beta \in I_{q,a}^t \) with \( B \alpha \equiv \beta \pmod{1} \). Then, for \( \beta = \frac{a}{q} + \eta, \eta \in \xi_q \), we write

\[ S_s(\alpha) = S \left( \frac{a}{q} + \eta \right) = \frac{\mu(q)}{\varphi(q)} T(\eta) + R(\eta, q, a). \]

Now, we have that

\[ R(\eta, q, a) = \sum_{Bk \leq Y} \Lambda(k) e \left( k \left( \frac{a}{q} + \eta \right) \right) - \frac{\mu(q)}{\varphi(q)} T(\eta) = \]
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\[
\sum_{b=1}^{q} e \left( \frac{ab}{q} \right) \sum_{k \equiv b (\mod q)}^{Bk \leq Y} \Lambda(k)e(k\eta) - \frac{\mu(q)}{\varphi(q)} T(\eta) + O(\log^2 Y) =
\]

\[
\sum_{b=1}^{q} e \left( \frac{ab}{q} \right) \frac{1}{\varphi(q)} \sum_{\chi} \chi(b) \sum_{Bk \leq Y} \Lambda(k)\chi(k)e(k\eta) - \frac{\mu(q)}{\varphi(q)} T(\eta) + O(\log^2 Y) =
\]

\[
\frac{1}{\varphi(q)} \sum_{\chi} \chi(a)\tau(\chi) W(\chi, \eta) + O(\log^2 Y).
\]

By partial summation, using the Siegel-Walfisz theorem, we have that

\[
W(\chi, \eta) \ll YL^{-6D},
\]

uniformly for \( q \leq L^D, \frac{a}{q} + \eta \in I_{q,a}' \). Thus, from the well-known estimate \( \tau(\chi) \ll q^{1/2} \), we obtain

\[
R(\eta, q, a) \ll YL^{-\frac{11}{2}D}, \quad (3)
\]

uniformly for \( \eta \in I_{q,a}', q \leq L^D, (a, q) = 1 \). Hence, we write

\[
\int_{\mathbb{H}} S_{\alpha}(\alpha)S_{\beta}(\alpha)e(-n\alpha)d\alpha = \frac{1}{B} \sum_{q \leq L^D} \sum_{a=1}^{q} \sum_{r=0}^{B-1} e\left(-\frac{n}{B} \left(\frac{a}{q} + r\right)\right) \times
\]

\[
\int_{I_{q,a}'} S_{\alpha}\left(\frac{1}{B} \left(\frac{a}{q} + r + \eta\right)\right)\left(\frac{\mu(q)}{\varphi(q)} T(\eta) + R(\eta, q, a)\right)e\left(-\frac{n}{B} \eta\right)d\eta \ll
\]

\[
\sum_{1} + \sum_{2} + \sum_{3},
\]

where

\[
\sum_{1} = \frac{1}{B} \sum_{q \leq L^D} \frac{\mu(q)}{\varphi(q)} \sum_{a=1}^{q} \sum_{r=0}^{B-1} e\left(-\frac{n}{B} \left(\frac{a}{q} + r\right)\right) \times
\]

\[
\int_{-\frac{1}{2}}^{\frac{1}{2}} S_{\alpha}\left(\frac{1}{B} \left(\frac{a}{q} + r + \eta\right)\right)T(\eta)e\left(-\frac{n}{B} \eta\right)d\eta,
\]

\[
\sum_{2} = \frac{1}{B} \sum_{q \leq L^D} \frac{\mu(q)}{\varphi(q)} \sum_{a=1}^{q} \sum_{r=0}^{B-1} e\left(-\frac{n}{B} \left(\frac{a}{q} + r\right)\right) \times
\]

\[
\sum_{3}.
\]
\[
\int_{\xi_q}^{1/2} S_q\left(\frac{1}{2} - \frac{n}{\eta} \right) T(\eta)e\left(- \frac{n}{\eta} \right) d\eta,
\]

\[
\sum \frac{1}{B} \sum_{q \leq L\varphi(q)}^B \sum_{a=1}^{q-1} \sum_{r=0}^{B-1} e\left(- \frac{n}{B} \left( \frac{a}{q} + r \right) \right) \times
\]

\[
\int_{\xi_q} \frac{S_q\left(1 - \frac{n}{\eta} \right) R(\eta, q, a)e\left(- \frac{n}{\eta} \right) d\eta. \right)
\]

Recalling that

\[
\frac{1}{B} \sum_{r=0}^{B-1} \varepsilon_r(\varphi(a)) = \begin{cases} 1 & \text{if } B \mid \varphi(a), \\ 0 & \text{if } B \not\mid \varphi(a), \end{cases}
\]

we have

\[
\sum = \sum_{q \leq L\varphi(q)}^B \sum_{N-Y < \varphi(a) \leq N} c_\varphi(\varphi(a)) \varphi(\varphi(a)) \Lambda(\varphi(a)) \times
\]

\[
\frac{1}{B} \sum_{r=0}^{B-1} \varepsilon_r(\varphi(a)) \sum_{N-Y < \varphi(a) \leq N} \int_{-\delta}^{\delta} e\left(\varphi(a) + Bk - n \right) d\eta\]

\[
= \sum_{q \leq L\varphi(q)}^B \sum_{N-Y < \varphi(a) \leq N} c_\varphi(\varphi(a)) \varphi(\varphi(a)) \Lambda(\varphi(a)). \quad (4)
\]

Using the property \(c_\varphi(m) = \sum_{d|m} d\mu\left(\frac{q}{d}\right)\), we write

\[
\sum_{N-Y < \varphi(a) \leq N} c_\varphi(\varphi(a)) \varphi(\varphi(a)) \Lambda(\varphi(a)) = \sum_{d|q} d\mu\left(\frac{q}{d}\right) \sum_{N-Y < \varphi(a) \leq N} \Lambda(\varphi(a)).
\]

We observe that, since \((A, B) = 1\) and \((AB, n) = 1\), the linear congruence \(a \equiv n \pmod{Bd}\) has a solution if and only if \((A, d) = 1\) and, if it exists, this solution is unique. Then, from Lemma 1 we obtain

\[
\frac{Y}{A} \sum_{d|q} \frac{d}{\varphi(d)B^{\mu\left(\frac{q}{d}\right)}} + O_{\varepsilon, \phi, \phi}(d) Y A^{-1} L^{-G} =
\]
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\[ \frac{Y}{A \varphi(B)} \sum_{d | q \ (d, A \varphi) = 1} \frac{\varphi(d, B)}{(d, B)} \frac{d}{\varphi(d)} \mu \left( \frac{q}{d} \right) + O_{\varepsilon, \eta, \omega}(q d(q) Y A^{-1} L^{-C}). \]  \hspace{1cm} (5)

For every square-free \( q \), we have

\[ \sum_{d | q \ (d, A \varphi) = 1} \varphi(d, B) \frac{d}{\varphi(d)} \mu \left( \frac{q}{d} \right) = \frac{c_\omega(n)}{\varphi(q)} f_{\lambda, \mu}(q), \]  \hspace{1cm} (6)

where \( f_{\lambda, \mu}(q) = \mu(q, B) \mu(q, A) \varphi(q, A) \) (see Lemma 2).

Since \( T(\eta) \ll \min(\sqrt{YB^{-1}}, \| \eta \|^{-1}) \), using the Cauchy-Schwarz inequality, we have that

\[ \left| \int_{\frac{4}{9}Y}^{2} S_A(B^{-1}(a/q + r + \eta)) T(\eta) e(-nB^{-1} \eta) d\eta \right| \leq \left| \int_{\frac{4}{9}Y}^{2} \left| S_A(B^{-1}(a/q + r + \eta)) \right|^2 d\eta \right|^{1/2} \times \left| \int_{\frac{4}{9}Y}^{2} \| \eta \|^{-2} d\eta \right|^{1/2} \ll q^{1/2} Y L^{1-2D}. \]

Analogously, from (3) we have that

\[ \int_{q} S_A(B^{-1}(a/q + r + \eta)) R(\eta, q, a) e(-nB^{-1} \eta) d\eta \ll q^{-1/2} Y L^{1-2D}. \]

Therefore, the contribution of \( \sum \frac{1}{2} \) and \( \sum \frac{3}{3} \) to \( \sum \) is

\[ \ll_{\lambda, \mu} H Y^2 \max(L^{2-D}, L^{2-4D}) \ll H Y^2 L^{-C}. \]  \hspace{1cm} (7)

Recalling that \( \frac{B}{\varphi(B)} = \prod_{p | B} \left( 1 + \frac{1}{p-1} \right) \), from (4)-(7) it follows that

\[ \sum_{N \in \mathcal{M}_B} \ll \sum_{N < n \leq N+H} \left| \sum_{\omega \in A} - \frac{Y}{AB} \Theta(n) \right|^2 + O(HY^2 L^{-C}) \ll \]

\[ \sum_{N < n \leq N+H} \left| \frac{Y}{AB} \left( \frac{B}{\varphi(B)} \sum_{q \leq L} \frac{\mu(q)}{\varphi(q)^2} f_{\lambda, \mu}(q) c_q(n) - \Theta(n) \right) \right|^2 + \]
The contribution of the first $O$-term is

$$\ll HY^2 L^{3-D-2G} \ll HY^2 L^{-C},$$

provided $2G > 3D + C$. Then, from the proof of Lemma 2, it is easy to see that

$$\sum_{N<n\leq N+H, n\in A} \left| \frac{B}{\varphi(B)} \sum_{q \leq LD} \frac{\mu(q)}{\varphi(q)} \mathcal{f}_{d,\omega}(q) c_q(n) - \mathcal{E}(n) \right|^2 \ll H L^{2-D} \ll H L^{-C}.$$

Thus, we conclude that (1) is proved.

**Proof of (2).** By the Cauchy-Schwarz inequality, Parseval's identity, the Brun-Titchmarsh inequality and the Lemma 3 we have that

$$\sum_{m \in A} \sum_{N<n\leq N+H, n\in A} \int_{m_B} S_\beta(\xi)S_\beta(\xi)e(-n\xi) d\xi \int_{m_B} S_\alpha(\alpha)S_\alpha(\alpha)e(n\alpha) d\alpha \ll \int_{m_B} |S_\beta(\xi)|^2 \left( \int_{m_B} |S_\alpha(\alpha)|^2 \right) d\xi \ll \left( \int_{m_B} |S_\beta(\xi)|^2 \right)^{\frac{1}{2}} \left( \int_{m_B} |S_\alpha(\alpha)|^2 \right)^{\frac{1}{2}} \ll \left( \int_{m_B} |S_\beta(\xi)|^2 \right)^{\frac{1}{2}} \left( \int_{m_B} |S_\alpha(\alpha)|^2 \right)^{\frac{1}{2}} \ll \sum_{s=1}^{A} \left( \int_{m_s} |S_\beta(\xi)|^2 \right)^{\frac{1}{2}} \left( \int_{m_s} |S_\alpha(\alpha)|^2 \right)^{\frac{1}{2}},$$

where $m_s = m \cap J_s$, with $J_s = \left[ \frac{s-1}{A}, \frac{s}{A} \right]$, $s = 1, ..., A.$
Hence, (2) is proved whenever
\[ \int_{m_0 \cap (\xi - \frac{1}{H}, \xi + \frac{1}{H})} |S(\beta)|^2 \, d\beta \ll Y L^{-2C-3}, \quad s = 1, \ldots, A \quad (8) \]
for \( Y_0(C, \varepsilon) \leq Y, \quad Y^{1/3 + \varepsilon} \leq H, \) uniformly for \( \xi \in [0, 1]. \)

Since, for \( \frac{a}{q} \neq \frac{a'}{q'} \) and \( q, q' \leq Q, \) we have that
\[ \left| \frac{a}{q} - \frac{a'}{q'} \right| \geq \frac{1}{Q^2} = \frac{4}{H}, \]
then there are at most two punctured arcs \( I_{q,a}^{\nu}, \) with \( q \leq Q \) and \( (a, q) = 1 \)
(see Lemma 4) which intersect \( (\xi - 1/H, \xi + 1/H). \) Then, in order to establish (8) it suffices to show that
\[ \max_{\substack{4 \leq q \leq Q \\atop (a, q) = 1}} \int_{I_{q,a}^{\nu}} |S(\beta)|^2 \, d\beta \ll Y L^{-2C-3}. \]

But this is given by Lemma 4. Hence, (2) is proved and the proof of Theorems 1 and 2 is complete.

Acknowledgement. The author is very grateful to Professor A. Perelli for his kind encouragement.

References


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Recibido: 9 de Febrero de 1996