

## *Approximation of Almost Periodic Functions by Convolution Type Operators*

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**ABSTRACT.** For  $S^p$ - and  $S^*$ -almost periodic functions  $f$  the convolution type operators  $L_\mu f$  are considered. The rates of convergence of  $L_\mu f(x)$  to  $f(x)$  at the Lebesgue or Lebesgue-Denjoy points  $x$  of  $f$  are estimated.

### 1. PRELIMINARIES

Let  $L_{loc}^p$  ( $1 \leq p < \infty$ ) be the class of all measurable complex-valued functions Lebesgue-integrable with  $p$ -th power on each finite interval and let  $D_{loc}^*$  be the set of all complex-valued functions integrable in the Denjoy-Perron sense on each finite interval. Denote by  $S^p$  and by  $S^*$  the spaces of all functions  $f \in L_{loc}^p$  and  $f \in D_{loc}^*$  which are  $S^p$ -almost periodic and  $S^*$ -almost periodic, respectively, with the norms

$$\|f\|_{S^p} := \sup_{-\infty < v < \infty} \left( \int_v^{v+1} |f(t)|^p dt \right)^{1/p}$$

and

$$\|f\|_{S^*} := \sup_{-\infty < v < \infty} \left( \sup_{0 \leq u \leq 1} \left| \int_v^{v+u} f(t) dt \right| \right).$$

Write  $S = S^1$  and use the symbol  $B$  for the space of all complex-valued functions  $f$  almost periodic in the Bohr sense, i.e. uniformly almost periodic, with the norm

$$\|f\|_B := \sup_{-\infty < v < \infty} |f(v)|.$$

The theory of Bohr's and  $S^p$ -almost periodic functions is given in [6]. Some properties of  $S^*$ -almost periodic functions can be found e.g. in [7], [8].

Let  $E$  be a set of positive numbers, having the accumulation point at infinity. Introduce the convolution type operators  $L_\mu$  ( $\mu \in E$ ), defined for functions  $f \in S$  or  $f \in S^*$  by the improper Denjoy-Perron integral

$$L_\mu f(x) := (f * \psi_\mu)(x) \equiv \int_{-\infty}^{-\infty} f(x-t) \psi_\mu(t) dt \quad (x \in R := (-\infty, \infty)), \quad (1)$$

where  $\psi_\mu$  are measurable (complex-valued) functions satisfying some additional assumptions. In particular, if  $f \in S^p$  with some  $p \geq 1$  and if  $\psi_\mu$  is Lebesgue-integrable on  $R$  (in symbols  $\psi_\mu \in L$ ), then  $L_\mu f$  is of class  $S^p$ . If  $f \in S^p$  ( $p > 1$ ),  $\psi_\mu \in L_{loc}^q$  (where  $\frac{1}{q} = 1 - \frac{1}{p}$ ) and

$$\|\psi_\mu\|_q := \sum_{k=-\infty}^{\infty} \left( \int_k^{k+1} |\psi_\mu(t)|^q dt \right)^{1/q} < \infty,$$

then  $L_\mu f$  is uniformly almost periodic; the same is also true if  $f \in S$ ,  $\psi_\mu \in L_{loc}^\infty$  (i.e.  $\psi_\mu$  is measurable and essentially bounded on each finite interval) and if

$$\|\psi_\mu\|_\infty := \sum_{k=-\infty}^{\infty} \operatorname{ess\,sup}_{k \leq t \leq k+1} |\psi_\mu(t)| < \infty.$$

In the case when  $f \in S^*$ , the assumptions

$$\|\psi_\mu\|_\infty < \infty \text{ and } \text{var}_{-\infty < t < \infty} \psi_\mu(t) < \infty$$

imply the uniform almost periodicity of  $L_\mu f$ , too (see [8], [9]).

In this paper, letting  $\mu \rightarrow \infty$ , we present some estimates for the rate of convergence of  $L_\mu f$  at the Lebesgue or Lebesgue-Denjoy points  $x$  of  $f$ . As a measure of deviation of  $L_\mu f(x)$  from  $f(x)$  we take the quantities

$$w_x(h; f) := \frac{1}{h} \int_0^h |\varphi_x(t)| dt \quad \text{if } f \in S,$$

$$w_x^*(h; f) := \sup_{0 < v \leq h} \frac{1}{v} \left| \int_0^v \varphi_x(t) dt \right| \quad \text{if } f \in S^*,$$

where  $h > 0$  and  $\varphi_x(t) := f(x+t) + f(x-t) - 2f(x)$ . For  $f \in S$  we also use the quantity

$$\bar{w}_x(h; f) := \sup_{0 < v \leq h} w_x(v; f).$$

Clearly,  $w_x(h; f) < \infty$  for all  $x$  and  $h > 0$ . In view of the well-known Lebesgue theorem and the fundamental properties of the Denjoy-Perron integral [5], for almost every  $x$ ,

$$\lim_{h \rightarrow 0+} w_x(h; f) = \lim_{h \rightarrow 0+} \bar{w}_x(h; f) = 0 \text{ and } \lim_{h \rightarrow 0+} w_x^*(h; f) = 0$$

(we call these  $x$  the Lebesgue and the Lebesgue-Denjoy points of  $f$ , respectively). Further,  $\bar{w}_x(h; f)$  and  $w_x^*(h; f)$  are non-decreasing functions of  $h$  on  $(0, \infty)$ , provided they are finite at  $x$ . The so-called local integral modulus  $\bar{w}_x(h; f)$  (in a slightly different form) was first used in [1] to obtain the quantitative version of the known Fejér-Lebesgue theorem.

For  $f \in S^p$  ( $p > 1$ ) we introduce also the quantities

$$w_x(h; f)_p := \left( \frac{1}{h} \int_0^h |\varphi_x(t)|^p dt \right)^{1/p} \quad (h > 0),$$

which have the properties similar to that of  $w_x(h; f)$ .

Throughout, the integral part of a real number  $a$  is denoted by  $[a]$ . The symbol  $\zeta(s)$ ,  $s > 1$ , means the well-known Riemann zeta function.

## 2. MAIN RESULTS

Consider operators  $L_\mu$  defined by (1), in which  $\psi_\mu$  are even measurable functions such that  $\|\psi_\mu\|_\infty < \infty$  or  $\|\psi_\mu\|_q < \infty$  with some  $q > 1$  (clearly, this implies that  $\psi_\mu$  are Lebesgue-integrable on  $R$ ).

**Theorem 1.** *Suppose that  $\|\psi_\mu\|_\infty < \infty$ ,*

$$\int_{-\infty}^{\infty} \psi_\mu(t) dt = 1 \text{ for all } \mu \in E \quad (2)$$

*and that there exist positive numbers  $\sigma, \alpha_\mu$  such that*

$$|\psi_\mu(t)| \leq \alpha_\mu t^{-\sigma} \text{ for a.e. } t \in (0, 1] \text{ and all } \mu \in E. \quad (3)$$

*If  $f \in S$ , then for every real  $x$ ,*

$$\begin{aligned} |L_\mu f(x) - f(x)| \leq & 2(\|f\|_S + |f(x)|)(\alpha_\mu + \gamma_\mu) \\ & + \beta_\mu \delta_\mu w_x(\delta_\mu; f) + \sigma \alpha_\mu \int_{\delta_\mu}^1 t^{-\sigma} w_x(t; f) dt, \end{aligned} \quad (4)$$

*where  $\delta_\mu$  are arbitrary positive numbers not greater than 1 and*

$$\beta_\mu := \operatorname{ess\,sup}_{0 < t \leq \delta_\mu} |\psi_\mu(t)|, \quad \gamma_\mu := \sum_{k=1}^{\infty} \operatorname{ess\,sup}_{k \leq t \leq k+1} |\psi_\mu(t)|.$$

**Proof.** In view of our assumptions, the convolution (1) exists for all  $x$  as the ordinary Lebesgue integral and

$$|L_\mu f(x) - f(x)| \leq \left( \int_0^{\delta_\mu} + \int_{\delta_\mu}^1 + \int_1^\infty \right) |\varphi_x(t)\psi_\mu(t)| dt = I_1 + I_2 + I_3, \text{ say.}$$

Clearly,

$$I_1 \leq \beta_\mu \int_0^{\delta_\mu} |\varphi_x(t)| dt = \beta_\mu \delta_\mu w_x(\delta_\mu; f),$$

$$I_3 \leq \sum_{k=1}^\infty \text{ess sup}_{k \leq t \leq k+1} |\psi_\mu(t)| \int_k^{k+1} |\varphi_x(t)| dt \leq 2 \left( \|f\|_S + |f(x)| \right) \gamma_\mu.$$

Further, by (3) and partial integration,

$$\begin{aligned} I_2 &\leq \alpha_\mu \int_{\delta_\mu}^1 |\varphi_x(t)| t^{-\sigma} dt = \alpha_\mu \int_{\delta_\mu}^1 \left( \int_0^t |\varphi_x(u)| du \right)' t^{-\sigma} dt \\ &\leq \alpha_\mu \left\{ w_x(1; f) + \sigma \int_{\delta_\mu}^1 t^{-\sigma} w_x(t; f) dt \right\}. \end{aligned}$$

Collecting the results and observing that  $w_x(1; f) \leq 2(\|f\|_S + |f(x)|)$  we get (4), immediately.

**Remark 1.** Assuming that  $\bar{w}_x(1; f) < \infty$ , one can easily verify that

$$\int_{\delta_\mu}^1 t^{-\sigma} \bar{w}_x(t; f) dt \leq \tau(\sigma) \sum_{k=1}^m k^{\sigma-2} \bar{w}_x\left(\frac{1}{k}; f\right),$$

where  $m := [1/\delta_\mu]$ ,  $\tau(\sigma) := \max\{1, 2^{\sigma-2}\}$ . Also, if  $\sigma \geq 1$ ,

$$\begin{aligned} \bar{w}_x(\delta_\mu; f) &\leq \bar{w}_x\left(\frac{1}{m}; f\right) \leq \frac{\sigma+1}{m^{\sigma+1}} \sum_{k=1}^m k^\sigma \bar{w}_x\left(\frac{1}{m}; f\right) \\ &\leq \frac{\sigma+1}{(m+1)^{\sigma-1}} 2^{\sigma-1} \sum_{k=1}^m k^{\sigma-2} \bar{w}_x\left(\frac{1}{k}; f\right) \\ &\leq 2^{\sigma-1} (\sigma+1) \delta_\mu^{\sigma-1} \sum_{k=1}^m k^{\sigma-2} \bar{w}_x\left(\frac{1}{k}; f\right). \end{aligned}$$

Consequently, under assumptions of Theorem 1 (with  $\sigma \geq 1$ ) we have

$$|L_\mu f(x) - f(x)| \leq 2(\|f\|_S + |f(x)|)(\alpha_\mu + \gamma_\mu) + c_\mu(\sigma) \sum_{k=1}^m k^{\sigma-2} \bar{w}_x\left(\frac{1}{k}; f\right),$$

where  $c_\mu(\sigma) = 2^{\sigma-1}(\sigma+1)\beta_\mu\delta_\mu^\sigma + \sigma\tau(\sigma)\alpha_\mu$ . In the case when  $\sigma = 1, 2$  or  $3$ , a direct calculation shows that the term  $2^{\sigma-1}$  in  $c_\mu(\sigma)$  may be omitted.

Let us note that Theorem 1 remains valid for functions  $f$  of class  $S^p$  with  $p > 1$ , because  $S^p \subset S$ . Nevertheless, in this case, the argumentation similar to that of the proof of Theorem 1 leads to

**Theorem 2.** *Let  $f \in S^p$  ( $p > 1$ ) and let  $\|\psi_\mu\|_q < \infty$  for all  $\mu \in E$ , where  $q = p/(p-1)$ . Suppose, moreover, that conditions (2) and (3) are satisfied. Then, for every  $x \in R$ ,*

$$|L_\mu f(x) - f(x)| \leq 2(\|f\|_{S^p} + |f(x)|)(\alpha_\mu + \gamma_{\mu,q}) + \beta_{\mu,q} \delta_\mu^{1/p} w_x(\delta_\mu; f)_p + \sigma \alpha_\mu \int_{\delta_\mu}^1 t^{-\sigma} w_x(t; f)_p dt,$$

where  $0 < \delta_\mu \leq 1$ ,

$$\beta_{\mu,q} := \left( \int_0^{\delta_\mu} |\psi_\mu(t)|^q dt \right)^{1/q}, \quad \gamma_{\mu,q} := \sum_{k=1}^{\infty} \left( \int_k^{k+1} |\psi_\mu(t)|^q dt \right)^{1/q}.$$

The corresponding result for almost periodic functions integrable in the Denjoy-Perron sense can be stated as follows.

**Theorem 3.** *Let  $\|\psi_\mu\|_\infty < \infty$ ,  $\text{var}_{-\infty < t < \infty} \psi_\mu(t) < \infty$  for all  $\mu \in E$  and let condition (2) be satisfied. Assume, moreover, that  $\psi_\mu$  are absolutely continuous on  $(0, 1]$  and that*

$$|\psi'_\mu(t)| \leq \alpha_\mu^* t^{-\rho} \text{ for a.e. } t \in (0, 1] \text{ and all } \mu \in E, \tag{5}$$

$\rho, \alpha_\mu^*$  being some positive numbers. If  $f \in S^*$  and if  $w_x^*(1; f) < \infty$  then

$$|L_\mu f(x) - f(x)| \leq 2(\|f\|_{S^*} + |f(x)|)\gamma_\mu^* + \frac{1}{2}\beta_\mu^* \delta_\mu^2 w_x^*(\delta_\mu; f) + \alpha_\mu^* \int_{\delta_\mu}^1 t^{-\rho+1} w_x^*(t; f) dt,$$

where

$$\beta_\mu^* := \operatorname{ess\,sup}_{0 < t \leq \delta_\mu} |\psi'_\mu(t)|, \quad \gamma_\mu^* := 2\gamma_\mu + \operatorname{var}_{1 \leq t < \infty} \psi_\mu(t),$$

$\delta_\mu$  and  $\gamma_\mu$  have the same meaning as in Theorem 1.

**Proof.** In view of (1) and (2),

$$L_\mu f(x) - f(x) = \left( \int_0^1 + \int_1^{\rightarrow \infty} \right) \varphi_x(t) \psi_\mu(t) dt = J_1 + J_2, \text{ say.}$$

Applying the known inequalities for the Denjoy-Perron integral ([5] p. 45, or [8] p. 187) we obtain

$$\begin{aligned} |J_2| &= \left| \sum_{k=1}^{\infty} \int_k^{k+1} \varphi_x(t) \psi_\mu(t) dt \right| \\ &\leq \sum_{k=1}^{\infty} \left( \sup_{k \leq t \leq k+1} |\psi_\mu(t)| + \operatorname{var}_{k \leq t \leq k+1} \psi_\mu(t) \right) \max_{k \leq \xi \leq k+1} \left| \int_k^\xi \varphi_x(t) dt \right| \\ &\leq 2 \left( \gamma_\mu + \operatorname{var}_{1 \leq t < \infty} \psi_\mu(t) \right) (\|f\|_{S^*} + |f(x)|). \end{aligned}$$

Further, putting

$$\Phi_x(t) := \int_0^t \varphi_x(u) du$$

and integrating by parts ([5] p. 42) we get

$$\begin{aligned}
 |J_1| &= \left| \Phi_x(1)\psi_\mu(1) - \int_0^1 \Phi_x(t)\psi'_\mu(t)dt \right| \\
 &\leq |\Phi_x(1)| |\psi_\mu(1)| + \left( \int_0^{\delta_\mu} + \int_{\delta_\mu}^1 \right) t w_x^*(t; f) |\psi'_\mu(t)| dt.
 \end{aligned}$$

Hence, assumption (5) and the obvious inequalities

$$|\psi_\mu(1)| \leq \gamma_\mu, \quad |\Phi_x(1)| \leq 2(\|f\|_{S^*} + |f(x)|)$$

give

$$|J_1| \leq 2\gamma_\mu(\|f\|_{S^*} + |f(x)|) + \frac{1}{2}\beta_\mu^* \delta_\mu^2 w_x^*(\delta_\mu; f) + \alpha_\mu^* \int_{\delta_\mu}^1 t^{-\rho+1} w_x^*(t; f) dt.$$

Collecting the results we get the desired assertion.

**Remark 2.** In the same way as in Remark 1, the estimate given in Theorem 3 can be stated in the form

$$|L_\mu f(x) - f(x)| \leq 2(\|f\|_{S^*} + |f(x)|)\gamma_\mu^* + c_\mu^*(\rho) \sum_{k=1}^m k^{\rho-3} w_x^*\left(\frac{1}{k}; f\right),$$

where  $m = [1/\delta_\mu]$ ,  $c_\mu^*(\rho) = 2^{\rho-3}(\rho+1)\beta_\mu^* \delta_\mu^\rho + \alpha_\mu^* \max\{1, 2^{\rho-3}\}$ , provided that  $\rho \geq 2$ .

Now, denoting by  $Y$  the space  $B$ ,  $S^p$  ( $p \geq 1$ ) or  $S^*$ , let us define the modulus of smoothness of  $f \in Y$  with respect to the norm of  $Y$  by

$$\omega_2(h; f)_Y := \sup_{0 \leq t \leq h} \|f(\cdot + t) + f(\cdot - t) - 2f(\cdot)\|_Y \quad (h \geq 0).$$

Clearly, if  $f \in B$  then, for all  $x \in R$  and  $h > 0$ ,

$$w_x(h; f) \leq \omega_2(h; f)_B.$$

In case  $f \in S^p$  we have

$$\sup_{-\infty < v < \infty} \left( \int_v^{v+1} (w_x(h; f))^p dx \right)^{1/p} \leq \omega_2(h; f)_{S^p} \quad (h > 0),$$

by the generalized Minkowski inequality. These estimates and Theorem 1 together with Remark 1 lead to the following

**Corollary.** Let  $f \in Y$ , where  $Y = B$  or  $S^p$  ( $p \geq 1$ ), and let conditions (2), (3) with  $\sigma \geq 1$  be satisfied. Then, for all  $\mu \in E$ ,

$$\|L_\mu f - f\|_Y \leq 4(\alpha_\mu + \gamma_\mu)\|f\|_Y + c_\mu(\sigma) \sum_{k=1}^m k^{\sigma-2} \omega_2\left(\frac{1}{k}; f\right)_Y,$$

where  $m, \alpha_\mu, \gamma_\mu, c_\mu(\sigma)$  have the same meaning as in Theorem 1 and Remark 1.

For almost periodic functions integrable in the Denjoy-Perron sense a direct calculation gives

**Theorem 4.** Suppose that  $f \in S^*$  and that conditions (2) and (5) with  $\rho \geq 2$  are satisfied. Then, for all  $\mu \in E$ ,

$$\|L_\mu f - f\|_{S^*} \leq 4\gamma_\mu^* \|f\|_{S^*} + c_\mu^*(\rho) \sum_{k=1}^m k^{\rho-3} \omega_2\left(\frac{1}{k}; f\right)_{S^*},$$

where  $m, \gamma_\mu^*$  and  $c_\mu^*(\rho)$  have the same meaning as in Theorem 3 and Remark 2.

### 3. EXAMPLES

I. Let  $0 \leq \lambda \equiv \lambda(\mu) < \mu$  for  $\mu \in E = (0, \infty)$  and let  $\Psi_{\lambda, \mu}$  be the continuous functions defined for  $t \neq 0$  by the formula

$$\Psi_{\lambda, \mu}(t) := \frac{(4 \sin \frac{1}{4}(\mu - \lambda)t)^2 \sin \frac{1}{2}(\mu + \lambda)t}{\pi(\mu - \lambda)^2 t^3}.$$

Denote by  $L_{\lambda, \mu}$  the operators (1) with  $\psi_\mu = \Psi_{\lambda, \mu}$ . As is known ([3] p. 256), condition (2) is satisfied. Introducing the auxiliary function  $g_z(t) := (\sin zt)/t$  for  $t \neq 0$ ,  $g_z(0) = z$ , with a positive parameter  $z$ , we can write

$$\Psi_{\lambda, \mu}(t) = \frac{1}{\pi} a^{-2} g_a^2(t) g_b(t) \text{ with } a = \frac{\mu - \lambda}{4}, b = \frac{\mu + \lambda}{2}.$$

Since

$$|g_z(t)| \leq \frac{1}{t}, |g'_z(t)| = \left| \frac{zt \cos zt - \sin zt}{t^2} \right| \leq \frac{2z}{t} \text{ for } t > 0$$

and

$$|g_z(t)| \leq z, |g'_z(t)| \leq \frac{2}{3} z^3 t \text{ for } t \geq 0,$$

we have

$$|\Psi_{\lambda, \mu}(t)| \leq \frac{16}{\pi(\mu - \lambda)^2 t^3}, |\Psi'_{\lambda, \mu}(t)| \leq \frac{32\mu}{\pi(\mu - \lambda)^2 t^3} \text{ for } t > 0$$

and

$$|\Psi_{\lambda, \mu}(t)| \leq \frac{\mu + \lambda}{2}, |\Psi'_{\lambda, \mu}(t)| \leq \frac{(\mu + \lambda)^3 t}{8\pi} \text{ for } t \geq 0.$$

These inequalities ensure that for every  $f \in S^p$  ( $p \geq 1$ ) or  $f \in S^*$  the functions  $L_{\lambda, \mu} f$  are uniformly almost periodic. Moreover, under the assumption  $\mu - \lambda \geq \underline{1}$ , Theorems 1, 3, 4 apply with  $\sigma = 3$ ,  $\rho = 3$ ,

$$\delta_\mu = \frac{1}{\mu - \lambda}, \quad \alpha_\mu = \frac{16}{\pi(\mu - \lambda)^2}, \quad \beta_\mu \leq \frac{\mu + \lambda}{2}, \quad \gamma_\mu \leq \frac{16\zeta(3)}{\pi(\mu - \lambda)^2},$$

$$\alpha_\mu^* = \frac{32\mu}{\pi(\mu - \lambda)^2}, \quad \beta_\mu^* \leq \frac{(\mu + \lambda)^3}{8\pi(\mu - \lambda)}, \quad \gamma_\mu^* \leq \frac{32(1 + \mu)\zeta(3)}{\pi(\mu - \lambda)^2}.$$

Assuming additionally that  $\frac{\lambda}{\mu} \leq \theta < 1$  for all  $\mu > 0$ , we easily verify that the right-hand sides of the estimates given in Theorems 1 - 3 and Remarks 1, 2 converge to zero as  $\mu \rightarrow \infty$ , for almost every  $x$ . In particular, setting  $\lambda(\mu) = \frac{\mu}{2}$  we get for  $f \in S$  the result of [3] (Th. 5). Moreover, from Corollary it follows the estimate of  $\|L_{\lambda, \mu} f - f\|_{S^p}$  in terms of the modulus of smoothness of  $f \in S^p$ . Namely,

$$\|L_{\lambda, \mu} f - f\|_{S^p} \leq \frac{21(1 + \zeta(3))}{(\mu - \lambda)^2} \|f\|_{S^p} +$$

$$+ 2 \left( \frac{1 + \theta}{1 - \theta} + 8 \right) \frac{1}{(\mu - \lambda)^2} \sum_{k=1}^m k \omega_2 \left( \frac{1}{k}; f \right)_{S^p},$$

where  $m = [\mu - \lambda]$  (clearly, the right-hand side of this inequality converges to zero as  $\mu \rightarrow \infty$ ). Taking into account the integral modulus of continuity

$$\omega_1(h; f)_{S^p} := \sup_{0 \leq t \leq h} \|f(\cdot + t) - f(\cdot)\|_{S^p}$$

and applying its basic properties, we easily verify that for  $f \in S^p$  with  $\omega_1(1; f)_{S^p} \neq 0$  there holds the relation

$$\|L_{\lambda, \mu} f - f\|_{S^p} = \mathcal{O} \left( \omega_1 \left( \frac{1}{\mu - \lambda}; f \right)_{S^p} \right),$$

which is equivalent to Theorem 1 of [3]. Note, that the corresponding estimates for  $f \in S^*$  follow from Theorem 4.

II. The Bernstein integral operators  $Q_\mu \equiv L_\mu$  are defined by (1), in which  $\mu \in E = (0, \infty)$ ,  $\psi_\mu = G_\mu$  are continuous functions on  $R$  with values

$$G_\mu(t) := \frac{c(r)}{\mu^{2r-1}} \left( \frac{1}{t} \sin \frac{\mu t}{2r} \right)^{2r}, \quad c(r) := (2r)^{2r-1} / \int_{-\infty}^{\infty} \left( \frac{\sin v}{v} \right)^{2r} dv,$$

for  $t \neq 0$ , and  $r$  is a fixed positive integer (see [4]). It is easy to verify that Theorems 1, 3, 4 are true for  $\mu \geq 1$  with

$$\delta_\mu = 1/\mu, \quad \sigma = \rho = 2r,$$

$$\alpha_\mu = \frac{c(r)}{\mu^{2r-1}}, \quad \beta_\mu \leq \frac{c(r)}{(2r)^{2r}}, \quad \gamma_\mu \leq \frac{c(r)\zeta(2r)}{\mu^{2r-1}},$$

$$\alpha_\mu^* = \frac{2c(r)}{\mu^{2r-2}}, \quad \beta_\mu^* \leq \frac{4c(r)\mu^2}{3(2r)^{2r+1}}, \quad \gamma_\mu^* \leq \frac{4c(r)\zeta(2r)}{\mu^{2r-2}}.$$

For almost every  $x$ , the right-hand side of the estimate corresponding to Theorem 1 converges to zero as  $\mu \rightarrow \infty$ , provided that  $r \geq 1$ . The same relation for the estimate following from Theorem 3 needs the assumption  $r \geq 2$ .

Note, that for some classes of functions the above results cannot be essentially improved. To see this, let us fix a point  $x$  and let us consider the class  $\Omega_x$  of all functions  $f \in S$  such that  $w_x(h; f) \leq h$  for  $0 < h \leq 1$ . In view of Theorem 1, for every  $f \in \Omega_x$  and every  $\mu \geq 1$ ,

$$\begin{aligned} & |Q_\mu f(x) - f(x)| \\ & \leq c(r) \left\{ 2 \left( 1 + \zeta(2r) \right) (\|f\|_S + |f(x)|) + (2r)^{-2r} + \frac{2r}{2r-2} \right\} \frac{1}{\mu} \end{aligned}$$

whenever  $r \geq 2$ . On the other hand, the function  $\eta_x$  of period 2, defined by  $\eta_x(t) := |t - x|$  if  $|t - x| \leq 1$ , belongs to  $\Omega_x$  and, for  $\mu \geq \pi r$ ,

$$\begin{aligned} |Q_\mu \eta_x(x) - \eta_x(x)| &= \int_0^\infty (\eta_x(x+t) + \eta_x(x-t)) G_\mu(t) dt \geq 2 \int_0^1 t G_\mu(t) dt \\ &\geq \frac{2c(r)}{\mu^{2r-1}} \int_{1/\mu}^{\pi r/\mu} t^{-2r+1} \left( \sin \frac{\mu t}{2r} \right)^{2r} dt \geq \frac{2c(r)}{\mu^{2r-1}} \int_{1/\mu}^{\pi r/\mu} t^{-2r+1} \left( \frac{\mu t}{\pi r} \right)^{2r} dt \\ &= \frac{c(r)}{(\pi r)^{2r}} (\pi^2 r^2 - 1) \frac{1}{\mu}. \end{aligned}$$

III. Let us suppose that the Fourier series of a function  $f \in S$  is of the form

$$f(x) \sim \sum_{k=-\infty}^{\infty} A_k e^{i\lambda_k x} \text{ with } A_k := \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T f(t) e^{-i\lambda_k t} dt,$$

$$0 < \lambda_k < \lambda_{k+1} \text{ if } k \in N := \{1, 2, \dots\}, \lim_{k \rightarrow \infty} \lambda_k = \infty, \lambda_{-k} = -\lambda_k,$$

$|A_k| + |A_{-k}| > 0$ , and let us consider its partial sums

$$S_n f(x) := \sum_{|\lambda_k| \leq \lambda_n} A_k e^{i\lambda_k x} \quad (n \in N).$$

As is known ([6] p. 83 and [2] Lemma 2),  $S_n f$  can be represented in the form (1), in which  $\mu = n \in N$  and  $\psi_\mu = D_n$ , where

$$D_n(t) := \frac{2}{\pi(\lambda_{n+1} - \lambda_n)} t^{-2} \sin \frac{1}{2}(\lambda_{n+1} - \lambda_n)t \sin \frac{1}{2}(\lambda_{n+1} + \lambda_n)t$$

for  $t \neq 0$ . If  $\lambda_{n+1} - \lambda_n \geq d > 0$ , where  $d$  is independent of  $n$ , then Theorem 1 gives the estimate

$$|S_n f(x) - f(x)| \leq 2 \left( \frac{\pi}{3} + \frac{2}{\pi} \right) (\|f\|_S + |f(x)|) \delta_n + \frac{1}{2\pi} (\lambda_{n+1} + \lambda_n) \delta_n w_x(\delta_n; f) + \frac{4}{\pi} \delta_n \int_{\delta_n}^1 t^{-2} w_x(t; f) dt \quad (6)$$

with  $\delta_n = d(\lambda_{n+1} - \lambda_n)^{-1}$ .

Assume that the Fourier series of  $f \in S$  is a lacunary series, i.e. there exists a positive number  $\theta < 1$  such that

$$\frac{\lambda_n}{\lambda_{n+1}} \leq \theta \quad \text{for all } n \in N.$$

Then inequality (6) holds with  $d = \lambda_1(1 - \theta)$ . Letting in this inequality  $n \rightarrow \infty$  and observing that  $\delta_n \rightarrow 0$  we easily state that  $S_n f(x) \rightarrow f(x)$  at every Lebesgue point  $x$  of the function  $f$ . Thus, from (6) it follows Theorem 2(1°) of [2], in a sharpened form.

If the Fourier exponents of  $f \in S$  satisfy the conditions

$$\lambda_{n+1} - \lambda_n \rightarrow \infty \quad \text{and} \quad \frac{\lambda_n}{\lambda_{n+1}} \rightarrow 1 \quad \text{as } n \rightarrow \infty,$$

then estimate (6) ensures that  $S_n f(x) \rightarrow f(x)$  at a Lebesgue point  $x$  of  $f$ , provided that the additional assumption

$$\lim_{n \rightarrow \infty} w_x \left( \frac{1}{\lambda_{n+1} - \lambda_n}; f \right) \left( 1 - \frac{\lambda_n}{\lambda_{n+1}} \right)^{-1} = 0$$

is satisfied (cf. Th. 2(2°) in [2]).

Finally, let us note that at the point  $x$  of continuity of  $f$ ,

$$w_x(h; f) \leq 2\omega(x; h; f), \quad \text{where } \omega(x; h; f) := \sup_{0 \leq t \leq h} |f(x+t) - f(x)|.$$

In this case it is convenient to estimate the term  $I_1$  in the proof of Theorem 1 as follows:

$$I_1 \leq 2\omega(x; \delta_\mu; f)\vartheta_\mu, \text{ where } \vartheta_\mu := \int_0^\infty |\psi_\mu(t)|dt.$$

Hence, inequalities (4) and (6) remain valid with  $w_x(h; f)$  replaced by  $\omega(x; h; f)$ ; the term  $\beta_\mu \delta_\mu$  in (4) and the corresponding term  $\frac{1}{2\pi}(\lambda_{n+1} + \lambda_n)\delta_n$  in (6) may be replaced by  $2\vartheta_\mu$  and by

$$2 \int_0^\infty |D_n(t)|dt \leq \frac{4}{\pi} + \frac{2}{\pi} \log \frac{\lambda_{n+1} + \lambda_n}{\lambda_{n+1} - \lambda_n},$$

respectively. So, inequality (6) in this form contains also Theorem 2(2°) of [2].

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