Embedding of Real Varieties and their Subvarieties into Grassmannians

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ABSTRACT. Given a compact affine nonsingular real algebraic variety $X$ and a nonsingular subvariety $Z \subset X$ belonging to a large class of subvarieties, we show how to embed $X$ in a suitable Grassmannian so that $Z$ becomes the transverse intersection of the zeros of a section of the tautological bundle on the Grassmannian.

In [2] Bochnak and Kucharz prove the following characterization of a compact nonsingular real affine hypersurface $Z$ in a compact affine nonsingular real algebraic variety $X$: There is an algebraic embedding $f: X \to \mathbb{R}P^n$ (for some $n$) and a projective hyperplane $H \subset \mathbb{R}P^n$ transverse to $f(X)$ such that $H \cap f(X) = f(Z)$. This fact (or rather a closely related statement about strongly algebraic real line bundles) plays a crucial role in their construction of algebraic models $Y$ of a compact, connected, smooth manifold $M$ of dimensions $m \geq 3$ such that

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the algebraic homology elements in $H^1(Y, \mathbb{Z}/2) = H^1(M, \mathbb{Z}/2)$ form a prescribed subgroup $G \subset H^1(M, \mathbb{Z}/2)$. If we wish to extend this result to subgroups of $H^k(M, \mathbb{Z}/2)$ for $k > 1$ it seems desirable, as a first step, to extend the above characterization of hypersurfaces to subvarieties of higher codimension.

Let $G_{n,k}(R)$ denote the Grassmannian of $k$-planes in $R^n$. Let $\gamma_{n,k}$ denote the universal bundle over $G_{n,k}(R)$. For definitions and results concerning real varieties, strongly algebraic vector bundles etc. see [1].

**Theorem 1.** Let $X$ be a compact affine nonsingular real algebraic variety. Let $\zeta$ be a strongly algebraic real vector bundle over $X$ of rank $k$. Let $\sigma$ be a regular section of $\zeta$ transverse to the zero section. Let $Z = \sigma^{-1}(0)$. Then

(i) There exists a regular embedding $f : X \to G_{n,k}(R)$ for suitable $n$ such that $\zeta$ and $f^*(\gamma_{n,k})$ are isomorphic.

(ii) There exists a regular section $s$ of $\gamma_{n,k}$ such that $s$ is transversal to the zero section and $s^{-1}(0) \cap f(X) = f(Z)$ (the intersection $s^{-1}(0) \cap f(X)$ being transverse intersection).

**Proof.** We can assume that $X$ is a subvariety of real projective $q$ space $\mathbb{R}P^q$ for some $q$. By theorem 12.1.7 of [1] there is a regular map $g : X \to G_{\ell,k}(R)$ (for suitable $\ell$) such that $g^*(\gamma_{\ell,k})$ and $\zeta$ are isomorphic. Let $G_{\ell,k}(C)$ denote the Grassmannian of complex $k$-planes in $C^\ell$ and $\gamma_{\ell,k}$ the corresponding universal complex bundle. Let $X_C$ denote the complexification of $X$ in $CP^\ell$. Then $g$ extends to a regular map $\tilde{g} : U \to G_{\ell,k}(C)$ where $U \subset X_C$ is a Zariski open set containing $X$. We can assume $U$ and $\tilde{g}$ are defined over $R$. By resolution of singularities we can find a complex nonsingular subvariety $Y$ of some complex projective space $CP^m$ with $Y$ defined over $R$ and a regular map (defined over $R$) $\tau : Y \to X_C$ where $\tau$ is the composition of a sequence of blowings-up with real centers outside $U$ such that $\tilde{g} \circ \tau$ extends to a regular map on $Y$. Denote this extension by $h$. To simplify notation we identify $X$ with $\tau^{-1}(X)$. Then $h^*(\gamma_{\ell,k})$ is a bundle defined over $R$ and $h^*(\gamma_{\ell,k})|X$ is isomorphic to $\zeta \otimes C$.

Now, for $E \to M$ a holomorphic vector bundle of rank $k$ over the compact complex manifold $M$, let $H^0(M, E)$ denote the space of holomorphic sections. Denote the dimension of $H^0(M, E)$ by $n$. Let
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\( i_E(x) = \{ \text{sections vanishing at } x \} \). Assume that each fiber of \( E \) is generated by global sections. Then identifying \( H^0(M, E) \) with \( \mathbb{C}^n \) we see that \( i_E \) maps \( M \) to \( G_{n,n-k}(C) \cong G_{n,k}(C) \). If \( F \to M \) is a positive holomorphic fiber bundle then for \( p \) sufficiently large \( i_{E \otimes F^p} \) is an embedding of \( M \) into \( G_{n,k}(C) \) where, now, \( n = \dim_{\mathbb{C}} H^0(M, E \otimes F^p) \) and \( i_{E \otimes F^p}^*(\gamma_{n,k}) \) is isomorphic to the bundle \( E \otimes F^p \to M \). Apply this to \( E \to M \) replaced by \( h^*(\gamma_{n,k}^C) \) (so \( M \) is replaced by \( Y \)) and \( F \) replaced by \( \gamma_{m,1}^C | Y \). In this case \( i_{E \otimes F^p} \) is a regular map defined over \( R \). Abbreviating \( i_{E \otimes F^p} \) by \( i \), we can write

\[ i^*(\gamma_{n,k}^C) \cong h^*(\gamma_{n,k}^C) \otimes (\gamma_{m,1}^C | Y)^p \]

(as complex bundles). We now restrict both sides to \( X \) and obtain

\[ (i|X)^*(\gamma_{n,k}) \otimes C \cong (\zeta \otimes C) \otimes ((\gamma_{m,1}|X) \otimes C)^p \]

and hence

\[ (i|X)^*(\gamma_{n,k}) \cong \zeta \otimes (\gamma_{m,1}|X)^p . \]

We can assume \( p \) is even. Then \( (\gamma_{m,1}|X)^p \) is topologically trivial. Hence \( (i|X)^*(\gamma_{n,k}) \) is topologically and hence algebraically isomorphic to \( \zeta \). This completes the proof of (i) with \( f = i|X \).

To simplify notation we now identify \( X \) with \( f(X) \) and \( \zeta \) with \( \gamma_{n,k}|X \). Let \( s_1, \ldots, s_n \) be sections of \( \gamma_{n,k} \) (over \( G_{n,k}(R) \)) spanning the fiber at each point of \( G_{n,k}(R) \). Write \( \sigma = \Sigma \lambda_i(s_i|X) \) where \( \lambda_i \) are regular real-valued functions on \( X \). Let \( \bar{\lambda} \) be a regular extension of \( \lambda_i \) to \( G_{n,k}(R) \). Let \( \phi \) be a regular real-valued function on \( G_{n,k}(R) \) such that \( \phi^{-1}(0) = Z(= \sigma^{-1}(0)) \). For \( t = (t_1, \ldots, t_n) \), define \( \tau = \Sigma_{i=1}^n (\bar{\lambda}_i + t_i \phi^2) s_i \). We can find \( t \) (suitably small) so that \( \tau \) is transverse to the zero section, \( s_t^{-1}(0) \) is transverse to \( X \) and \( s_t^{-1}(0) \cap X = \sigma^{-1}(0)(= Z) \). This completes the proof of (ii).

References


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