Sequence Spaces Generated by Moduli of Smoothness

J. MUSIELAK and A. WASZAK

ABSTRACT. There are defined sequential moduli in the remainder form for real sequences. Properties of sequence spaces generated by means of the above moduli are investigated.

1. INTRODUCTION

In many problems of mathematical analysis, one of the important tools form moduli of continuity and smoothness and variations of a function. The modulus of continuity may be defined in spaces of continuous functions and in $L^p$-spaces. In [6] and [7] we transferred the notion of modulus of continuity to spaces of sequences, by the formula

$$\omega(x,r) = \sup_{n \geq r, i \geq m} |t_{m+i} - t_i|,$$

where $x = (t_i)_{i=0}^\infty$, $r = 0, 1, 2, \ldots$. We developed a theory of modular spaces of sequences generated by the modulus (see also [3]).

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In the present paper we transfer the definition of $L^p$-modulus to
the sequential case, introducing the remainder form of the sequential
modulus. Moreover, we replace the power $p$ by a sequence of $\varphi$-functions,
$\varphi = (\varphi_i)_{i=1}^{\infty}$, (for definition of $\varphi$-function see for instance [4], 1.9). There
are analysed structural properties of modular spaces generated by means
of the above notions. In a subsequent paper we shall show application
to problems of two modular convergence of sequences with aid of moduli
of smoothness and $\Phi$-variations and we shall derive some inequalities.

2. MODULUS OF SMOOTHNESS

We introduce the remainder form of the sequential modulus in the
space $X$ of all real sequences. Let $x = (t_i)_{i=0}^{\infty} \in X$, then we denote
$(x)_j = t_j$ and we write $(\tau_m x)_j = t_j$ for $j < m$ and $(\tau_m x)_j = t_{m+j}$ for
$j \geq m$ where $m,j = 0,1,2,\ldots$. The sequence $\tau_m x = ((\tau_m x)_j)_{j=0}^{\infty}$ is
called the $m$-translation of the sequence $x$ (see [6]). Let $\varphi = (\varphi_i)_{i=1}^{\infty}$
be a sequence of $\varphi$-functions. The remainder form of the sequential
$\varphi$-modulus of the sequence $x$ will be defined as

$$
\omega_{\varphi}(x,r) = \sup_{m \geq r} \sum_{i=1}^{\infty} \varphi_i(|(\tau_m x)_i - (x)_i|), \quad r = 0,1,2,\ldots
$$

Obviously, we have

$$
\omega_{\varphi}(x,r) = \sup_{m \geq r} \sum_{i=m}^{\infty} \varphi_i(|t_{m+i} - t_i|).
$$

For any two sequences $x$ and $y$ we have

$$
\omega_{\varphi}(x+y,r) \leq \omega_{\varphi}(2x,r) + \omega_{\varphi}(2y,r).
$$

Let $\Psi$ be a nonnegative, nondecreasing function of $u \geq 0$ such that
$\Psi(u) \to 0$ as $u \to 0+$, $\Psi(u)$ not vanishing identically, and let $(a_r)$ be
sequence of positive numbers with $a = \inf_{r \geq 0} a_r > 0$. We define the set

$$
X(\Psi) = \{x \in X : a_r \Psi(\omega_{\varphi}(\lambda x, r)) \to 0 \text{ as } r \to \infty \text{ for a } \lambda > 0\}.
$$
3. \(\varphi\)-FUNCTIONS AND THEIR PROPERTIES

We shall need the following conditions concerning the function \(\Psi\) and functions \(\varphi_i, i = 1, 2, \ldots\).

The function \(\Psi\) is said to satisfy the conditions \((\Delta_2)\) for small \(u\) (for all \(u\)), if there are \(u_0 > 0\) and \(K > 0\) such that \(\Psi(2u) \leq K\Psi(u)\) for all \(0 < u \leq u_0\) (for all \(u \geq 0\)).

This implies that for every \(u_1 > 0\) there exists \(K_1 > 0\) such that \(\Psi(2u) \leq K_1\Psi(u)\) for all \(0 < u \leq u_1\).

The sequence \(\varphi = (\varphi_i)_{i=1}^{\infty}\) will be said to satisfy the condition \((A)\), if for every \(\varepsilon > 0\) there exist \(A > 0\) and \(\alpha > 0\) such that for all \(0 \leq u \leq A\) for all \(i = 1, 2, \ldots\)

\[ \varphi_i(\alpha u) \leq \varepsilon \varphi_i(u). \]

The sequence \(\varphi = (\varphi_i)_{i=1}^{\infty}\) will be said to satisfy the condition \((A')\), if there exists an \(\alpha > 0\) such that for every \(u \geq 0\), for all \(i = 1, 2, \ldots\)

\[ 2\varphi_i(\alpha u) \leq \varphi_i(u). \]

Let us remark that if the functions \(\varphi_i\) are all \(s\)-convex with a fixed \(s \in (0, 1)\) then \(\varphi = (\varphi_i)_{i=1}^{\infty}\) satisfies both conditions \((A)\) and \((A')\), (for definition of \(s\)-convex function see e.g. [2], [4], [6]). A converse statement is not true. For example, taking

\[ \varphi_i(u) = \varphi(u) = 1 - \sqrt{1 + \frac{1}{\ln u}} \]

for \(0 < u < v_0\), with \(v_0\) sufficiently small, we see easily that \((A)\) is satisfied but \(\varphi\) is not equivalent to an \(s\)-convex function for \(0 < s \leq 1\).

We shall say that the function \(\Psi\) satisfies the condition \((B)\), if there exists a \(v > 0\) such that for every \(\delta > 0\) there is an \(\eta > 0\) satisfying the inequality \(\Psi(\eta u) \leq \delta \Psi(u)\) for any \(0 \leq u \leq v\).

The sequence \(\varphi = (\varphi_i)_{i=1}^{\infty}\) of \(\varphi\)-functions will be said to satisfy the condition \((C)\), if for every \(\eta > 0\) there exists an \(\varepsilon > 0\) such that for all \(u > 0\) and all indices \(i\), the inequality \(\varphi_i(u) < \varepsilon\) implies \(u < \eta\).
Let us remark that (C) implies that \( \varphi_i(u) > 0 \) if \( u > 0 \).

4. SPACE \( X(\Psi) \)

We give now some characteristic of the space \( X(\Psi) \) defined in 2, and we investigate the vector structure on \( X(\Psi) \).

**Theorem 1.** Let us suppose that \( \Psi \) satisfies the condition \( (\Delta_2) \) for small \( u \) and let the functions \( \varphi_i \) satisfy \( (\Delta_2) \) for all \( u \) with a constant \( K > 0 \) independent of \( i \). Then \( x \in X(\Psi) \) if and only if \( a_r \Psi(\omega(\lambda x, r)) \to 0 \) as \( r \to \infty \) for every \( \lambda > 0 \).

The easy proof will be omitted.

**Remark 1.** It is easy to verify that if \( \varphi_i \) satisfy \( (\Delta_2) \) for small \( u \) with \( K \) and \( u_0 \) independent of \( i \) and the sequence \( x \) is bounded, then the thesis of Theorem 1 is true.

**Theorem 2.** Let one of the following two conditions hold:

1°. \( \Psi \) satisfies \( (\Delta_2) \) for small \( u \),

2°. \( \varphi \) satisfies \( (A') \).

Then \( X(\Psi) \) is a vector space.

**Proof.** Supposing \( x, y \in X(\Psi) \) and applying the inequality \( \varphi(u + v) \leq \varphi(2u) + \varphi(2v) \), we obtain for \( x = (i_i), \ y = (s_i) \)

\[
\omega(2x, r) \leq \omega(2x, r) + \omega(2y, r)
\]

for every \( r > 0 \). Now, by the definition of \( X(\Psi) \) there exists a \( \lambda > 0 \) such that \( a_r \Psi(\omega(\lambda x, r)) \to 0 \) and \( a_r \Psi(\omega(\lambda y, r)) \to 0 \) as \( r \to \infty \). We
have
\[ a_r \Psi \left( \omega_\varphi \left( \frac{1}{2} \lambda (x + y), r \right) \right) \leq a_r \Psi [\omega_\varphi(\lambda x, r) + \omega_\varphi(\lambda y, r)] \leq \]
\[ \leq a_r \Psi (2\omega_\varphi(\lambda x, r)) + a_r \Psi (2\omega_\varphi(\lambda y, r)), \]
by monotonicity of the function \( \Psi \).

Now, let us suppose 1. By assumptions, there are constants \( M, \delta > 0 \) such that \( 0 < \Psi(u) \leq \delta \) implies \( u \leq M \). Since \( a_r \Psi(\omega_\varphi(\lambda x, r)) \to 0 \) as \( r \to \infty \) and \( a = \inf_{r \geq 0} a_r > 0 \), we have \( \Psi(\omega_\varphi(\lambda x, r)) \to 0 \) as \( r \to \infty \).

Hence there exists an \( r_1 > 0 \) such that \( \Psi(\omega_\varphi(\lambda x, r)) \leq \delta \) for \( r \geq r_1 \). Consequently, \( \omega_\varphi(\lambda x, r) \leq M \) for \( r \geq r_1 \). Similarly \( \omega_\varphi(\lambda y, r) \leq M \) for \( r \geq r_2 \) with some \( r_2 > 0 \), and we may suppose \( r_2 = r_1 \). Taking \( u_1 = M \), by 1 there is a \( K_1 > 0 \) such that \( \Psi(2\omega_\varphi(\lambda x, r)) \leq K_1 \Psi(\omega_\varphi(\lambda x, r)) \) and \( \Psi(2\omega_\varphi(\lambda y, r)) \leq K_1 \Psi(\omega_\varphi(\lambda y, r)) \) for \( r \geq r_1 \). Hence for \( r \geq r_1 \) we obtain
\[ a_r \Psi \left( \omega_\varphi \left( \frac{1}{2} \lambda (x + y), r \right) \right) \leq K_1 \left[ a_r \Psi(\omega_\varphi(\lambda x, r)) + a_r \Psi(\omega_\varphi(\lambda y, r)) \right] \to 0 \]
as \( r \to \infty \). Hence \( x + y \in X(\Psi) \).

Next, let us suppose 2. Then
\[ \omega_\varphi(\alpha \lambda x, r) = \sup_{m \geq r} \sum_{i=m}^{\infty} \varphi_1(\alpha \lambda |t_{i+m} - t_i|) \leq \]
\[ \leq \frac{1}{2} \sup_{m \geq r} \sum_{i=m}^{\infty} \varphi_1(\lambda |t_{i+m} - t_i|) = \frac{1}{2} \omega_\varphi(\lambda x, r) \]
and similarly
\[ \omega_\varphi(\alpha \lambda y, r) \leq \frac{1}{2} \omega_\varphi(\lambda y, r) \]
for \( r \geq 0, \lambda > 0 \).
Thus
\[ a_r \Psi \left( \omega \left( \frac{1}{2} \lambda (x + y), r \right) \right) \leq a_r \Psi (2\omega \left( \lambda \alpha x, r \right)) + a_r \Psi (2\omega \left( \lambda \alpha y, r \right)) \leq a_r \Psi (\omega (\lambda x, r)) + a_r \Psi (\omega (\lambda y, r)) \rightarrow 0 \]
as \( r \rightarrow \infty \) for sufficiently small \( \lambda > 0 \). Hence \( x + y \in X(\Psi) \). This proves the theorem.

5. MODULAR STRUCTURE ON \( X(\Psi) \)

For every \( a \in X \) we define the functional
\[
\zeta(a) = \sup \{ a_r \Psi (\omega (x, r)) \} = \sup \{ a_r \Psi \left[ \sup_{m \geq r} \sum_{i=m}^{\infty} \psi_i (|t_{i+m} - t_i|) \right] \}.
\]

**Theorem 3.** Let \( \varphi = (\varphi_i)_{i=1}^{\infty} \) and \( \Psi \) satisfy one of the following two conditions:

1° \( \Psi \) is concave,

2° functions \( \varphi_i \) are convex.

Then \( X(\Psi) \) is a vector space and \( \zeta \) is a pseudomodular in \( X \).

**Proof.** If \( \Psi \) is concave and \( \Psi (0) = 0 \) then \( \Psi \) satisfies the condition \((\Delta_2)\) for all \( u > 0 \), because \( \Psi (2u) \leq 2\Psi (u) \). Hence, by Theorem 2, \( X(\Psi) \) is a vector space. Moreover, if \( x, y \in X \), \( x = (t_i) \), \( y = (s_i) \), \( \alpha, \beta \geq 0 \), \( \alpha + \beta = 1 \), then
\[
\zeta(\alpha x + \beta y) \leq \sup_{r \geq 0} a_r \Psi \left[ \sup_{m \geq r} \sum_{i=m}^{\infty} \psi_i (|t_{i+m} - t_i| + \beta |s_{i+m} - s_i|) \right] \leq \zeta(x) + \zeta(y).
\]
Consequently, \( \zeta \) is a pseudomodular.
Now, let us suppose \( \varphi \) to be convex for \( i = 1, 2, \ldots \). Then \( \varphi = (\varphi_i)_{i=1}^{\infty} \) satisfies \((A')\) and so, by Theorem 2, \( X(\Psi) \) is a vector space. Moreover, with the same notation as above, we have

\[
\zeta(\alpha x + \beta y) \leq \sup_{r \geq 0} a_r \Psi \left[ \sup_{m \geq r} \sum_{i=m}^{\infty} \varphi_i(\alpha |t_{i+m} - t_i| + \beta |s_{i+m} - s_i|) \right] \leq \\
\leq \sup_{r \geq 0} a_r \Psi \left[ \sup_{m \geq r} \sum_{i=m}^{\infty} \varphi_i(|t_{i+m} - t_i|) \right] + \\
+ \sup_{r \geq 0} a_r \Psi \left[ \sup_{m \geq r} \sum_{i=m}^{\infty} \varphi_i(|s_{i+m} - s_i|) \right] = \zeta(x) + \zeta(y).
\]

Hence \( \zeta \) is a pseudomodular in \( X \).

As well-known, the pseudomodular \( \zeta \) defines an \( F \)-pseudonorm

\[
|x|_\zeta = \inf \left\{ u > 0 : \zeta \left( \frac{x}{u} \right) \leq u \right\}
\]

in the modular space

\[
X_\zeta = \{ x \in X : \zeta(\lambda x) \to 0 \text{ as } \lambda \to 0_+ \}
\]

(compare [5], [8]).

We shall investigate \( \zeta \) in case when \( \Psi \) is s-convex with \( 0 < s \leq 1 \).

**Remark 2.** Let \( \Psi \) be s-convex with \( 0 < s \leq 1 \) and let \( \varphi_i \) be convex for \( i = 1, 2, \ldots \). Then \( \zeta \) is an \( s \)-convex pseudomodular, i.e.

\[
\zeta(\alpha x + \beta y) \leq \alpha^s \zeta(x) + \beta^s \zeta(y)
\]

if \( \alpha, \beta \geq 0, \alpha^s + \beta^s \leq 1 \).

For proof, let us remark that by Theorem 3, \( \zeta \) is a pseudomodular. Moreover, taking \( x = (t_i), y = (s_i), \alpha, \beta \geq 0, \alpha^s + \beta^s \leq 1 \), we have \( \alpha + \beta \leq 1 \) and so
\[\varsigma(\alpha x + \beta y) \leq \sup_{r \geq 0} \alpha \sup_{m \geq r} \sum_{i=m}^{\infty} \varphi_i(|t_{i+m} - t_i|) + \beta \sup_{m \geq r} \sum_{i=m}^{\infty} \varphi_i(|s_{i+m} - s_i|) \leq \alpha^s \varsigma(x) + \beta^s \varsigma(y).\]

**Theorem 4.** Let the function $\Psi$ be increasing, continuous and $s$-convex and let the functions $\varphi_i$ be convex, $i = 1, 2, \ldots$, where $0 < s \leq 1$. Then the $s$-homogeneous pseudonorm

\[\|x\|_s^2 = \inf\{u > 0 : \varsigma\left(\frac{x}{u^{1/s}}\right) \leq 1\}\]

satisfies the following inequalities:

1° if $x \in X_\varsigma$, $\|x\|_s^2 < 1$, then

\[\|x\|_s^2 \geq \sup_r \left(\frac{\omega(x, r)}{\Psi^{-1}(1/a_r)}\right)^s,\]

2° if $x \in X_\varsigma$, $\|x\|_s^2 > 1$, then

\[\|x\|_s^2 \leq \sup_r \left(\frac{\omega(x, r)}{\Psi^{-1}(1/a_r)}\right)^s,\]

where $\Psi^{-1}$ is the inverse to $\Psi$.

**Proof.** Since, by Remark 2, $\varsigma$ is $s$-convex, so $\| \cdot \|_s$ is an homogeneous pseudonorm. Let $\|x\|_s^2 < u < 1$, then

\[a_r \Psi\left(\frac{1}{u^{1/s}} \sup_{m \geq r} \sum_{i=m}^{\infty} \varphi_i(|t_{i+m} - t_i|)\right) \leq 1\]

for all $r \geq 0$. Hence

\[\sup_{m \geq r} \sum_{i=m}^{\infty} \varphi_i(|t_{i+m} - t_i|) \leq u^{1/s} \Psi^{-1}(1/a_r),\]
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\[
\omega_\varphi(x, r) \leq u^{1/s} \Psi^{-1}(1/a_r),
\]
which gives the inequality 1°, when we take \( u \to ||x||^p \).

Now, if \( ||x||^p > u > 1 \), then we have
\[
\sup_r a_r \Psi \left( \frac{1}{u^{1/s}} \omega_\varphi(x, r) \right) > 1
\]
and we obtain the inequality 2° easily.

**Corollary.** By the assumptions of Theorem 4, if
\[
\sup_r \frac{\omega_\varphi(x, r)}{\Psi^{-1}(1/a_r)} = 1,
\]
then \( ||x||^p = 1 \).

Let \( \bar{c} \) be the space of all sequences \( x = (t_i)_{i=0}^{\infty} \) such that \( t_i = t_{i+1} \)
for \( i = 1, 2, \ldots \). There holds the following

**Remark 3.** Let us remark that if \( \Psi(u) > 0 \) for \( u > 0 \), then \( x \in \bar{c} \)
if and only if \( |x| = 0 \).

6. COMPLETENESS

Taking the assumptions of Theorem 2, we may consider the quotient spaces: \( \hat{X}_c = X_c/\bar{c} \) and \( \hat{X}(\Psi) = X(\Psi)/\bar{c} \), with elements \( \hat{x}, \ldots \) (see [1]).
The \( F \)-pseudonorms resp. \( s \)-convex pseudonorms may be defined by
\( ||\hat{x}||_c = |x|_c, ||\hat{x}||^p = ||x||^p \), where \( x \in \hat{x} \), respectively.

**Theorem 5.** Let \( \Psi \) be increasing, continuous and satisfying the condition (B). Let \( \varphi = (\varphi_i)_{i=1}^{\infty} \) satisfy conditions (A) and (C). Moreover, let at least one of the following two conditions hold:

1° \( \Psi \) is concave,

2° \( \varphi_i \) are convex.

Then \( \hat{X}_c \) is a Fréchet space with respect to the \( F \)-norm \( ||.||_c \).

**Proof.** Let \( (\hat{x}_n) \) be a Cauchy sequence in \( \hat{X}_c \), \( x_n \in \hat{x}_n, x_n = (t^n_i)_{i=0}^{\infty} \). Without loss of generality, we may suppose that \( t^n_1 = 0 \) for
$n = 1, 2, \ldots$. We denote by $\Psi_{-1}$ the inverse function to $\Psi$. Since $a = \inf_{r \geq 0} a_r > 0$, for every $\varepsilon > 0$ one can find an $N$ such that $|x_p - x_q| < a \Psi(\varepsilon)$ for $p, q > N$. By the definition of $|.|_c$, there exists $u_\varepsilon$ such that $0 < u_\varepsilon < a \Psi(\varepsilon)$ and $c\left(\frac{|x_p - x_q|}{u_\varepsilon}\right) \leq u_\varepsilon$ for $p, q > N$. Consequently,

$$a_r \Psi\left(\omega_\varphi\left(\frac{x_p - x_q}{u_\varepsilon}, r\right)\right) \leq u_\varepsilon$$

for $p, q > N$ and $r \geq 0$, whence

$$\omega_\varphi\left(\frac{x_p - x_q}{u_\varepsilon}, r\right) \leq \Psi_{-1}\left(\frac{u_\varepsilon}{a_r}\right) \leq \Psi_{-1}\left(\frac{u_\varepsilon}{a}\right) < \varepsilon$$

for $p, q > N$, $r \geq 0$. By the definition of $\omega_\varphi$, we obtain in particular

$$\sum_{i=m}^{s} \varphi_i\left(\frac{1}{u_\varepsilon}|t_{i+m}^p - t_{i+m}^q - t_i^p + t_i^q|\right) < \Psi_{-1}\left(\frac{u_\varepsilon}{a_r}\right) < \varepsilon \quad (1)$$

for $p, q > N$, $s \geq m$ and $i \geq m \geq r \geq 0$. By condition (C), for every $\eta > 0$ one can find an $\varepsilon > 0$ such that

$$\frac{1}{u_\varepsilon}|t_{i+m}^p - t_{i+m}^q - t_i^p + t_i^q| < \eta \quad (2)$$

for $p, q > N$, $i \geq m \geq 0$. Hence

$$|t_{i+m}^p - t_{i+m}^q| < |t_i^p - t_i^q| + \eta u_\varepsilon < |t_i^p - t_i^q| + \eta a \Psi(\varepsilon)$$

for $p, q > N$, $i \geq m \geq 0$. Since $t_0^i = 0$ for $n = 1, 2, \ldots$, the above inequalities imply $(t_i^p)_{p=1}^{\infty}$ to be Cauchy sequences for $i = 1, 2, \ldots$. Hence these sequences are convergent. Let us write $t_i = \lim_{n \to \infty} t_i^n$ for $i = 1, 2, \ldots$, $t_0 = 0$, $x = (t_i)_{i=0}^{\infty}$. Taking $q \to \infty$ in (1), we obtain

$$\sum_{i=m}^{s} \varphi_i\left(\frac{|t_{i+m}^p - t_{i+m}^q - t_i^p + t_i|}{u_\varepsilon}\right) \leq \Psi_{-1}\left(\frac{u_\varepsilon}{a_r}\right)$$
for $p > N$, $s \geq m \geq r \geq 0$. Again, taking $s \to \infty$, we get

$$\sum_{i=m}^{\infty} \varphi_i \left( \left| \frac{t_{i+m}^p - t_i^p + t_i}{a_r} \right| \right) \leq \Psi^{-1} \left( \frac{u_\varepsilon}{a_r} \right)$$

for $p > N$, $m \geq r \geq 0$. Thus,

$$\omega_p \left( \frac{x_p - x}{u_\varepsilon}, r \right) \leq \Psi^{-1} \left( \frac{u_\varepsilon}{a_r} \right)$$

for $p > N$, $r \geq 0$. Hence

$$a_r \Psi \left( \omega_p \left( \frac{x_p - x}{u_\varepsilon}, r \right) \right) \leq u_\varepsilon$$

(3)

for $p > N$ and $r \geq 0$.

We are going to prove that $x_p - x \in X_\varepsilon$ for large $p$, i.e. $\varepsilon(\lambda(x_p - x)) \to 0$ as $\lambda \to 0_+$. Let $\varepsilon > 0$ be fixed and let $N$ be chosen as above.

Let $p > N$. We have for $\lambda > 0$

$$\omega_p(\lambda(x_p - x), r) = \omega_p \left( \lambda u_\varepsilon \frac{x_p - x}{u_\varepsilon}, r \right) =$$

$$= \sup_{m \geq r} \sum_{i=m}^{\infty} \varphi_i \left( \left| \frac{t_{i+m}^p - t_i^p + t_i}{u_\varepsilon} \right| \right).$$

Taking $q \to \infty$ in (2) we obtain

$$\left| \frac{t_{i+m}^p - t_i^p - t_i}{u_\varepsilon} \right| \leq \eta$$
for $i \geq m \geq 0$. We apply the condition (A) with $\varepsilon$ in place of $\varepsilon$, $\lambda \leq \alpha/u_{\varepsilon}$, and we choose $\eta = A$. Then for $u = \frac{1}{u_{\varepsilon}}|t_{i+m}^p - t_i^p + t_i|$ we get

$$\phi_i\left(\lambda u_{\varepsilon} \frac{|t_{i+m}^p - t_i^p + t_i|}{u_{\varepsilon}}\right) \leq \varepsilon \phi_i\left(\frac{1}{u_{\varepsilon}}|t_{i+m}^p - t_i^p + t_i|\right),$$

for $p > N$, $i \geq m \geq 0$. Hence

$$\omega_{\varepsilon}(\lambda(x_p - x), r) \leq \varepsilon \sup_{m \geq r} \sum_{i=m}^{\infty} \phi_i\left(\frac{1}{u_{\varepsilon}}|t_{i+m}^p - t_i^p + t_i|\right) = \varepsilon \omega_{\varepsilon}\left(\frac{x_p - x}{u_{\varepsilon}}, r\right) \leq \varepsilon \varepsilon^{-1}\left(\frac{u_{\varepsilon}}{a_r}\right) \leq \varepsilon \cdot \varepsilon.$$

Hence for $0 < \lambda \leq \alpha/u_{\varepsilon}$ we have

$$\zeta(\lambda(x_p - x)) = \sup_{r \geq 0} a_r \Psi(\omega_{\varepsilon}(\lambda(x_p - x), r)) \leq \sup_{r \geq 0} a_r \Psi\left(\varepsilon \varepsilon^{-1}\left(\frac{u_{\varepsilon}}{a_r}\right)\right).$$

Now, we apply the condition (B) with $v = \Psi^{-1}\left(\frac{u_{\varepsilon}}{a_r}\right)$, $u = \Psi^{-1}\left(\frac{u_{\varepsilon}}{a_r}\right)$. Choosing $\delta > 0$ arbitrarily and taking $\epsilon = \eta$, we obtain

$$\Psi\left(\varepsilon \varepsilon^{-1}\left(\frac{u_{\varepsilon}}{a_r}\right)\right) \leq \delta \Psi\left(\varepsilon^{-1}\left(\frac{u_{\varepsilon}}{a_r}\right)\right) = \delta \frac{u_{\varepsilon}}{a_r}.$$

Consequently,

$$\zeta(\lambda(x_p - x)) \leq \sup_{r \geq 0} a_r \delta \frac{u_{\varepsilon}}{a_r} = \delta u_{\varepsilon} \text{ for } 0 < \lambda \leq \alpha/u_{\varepsilon}.$$

Since $u_{\varepsilon}$ is fixed, this implies $\zeta(\lambda(x_p - x)) \rightarrow 0$ as $\lambda \rightarrow 0_+$. Hence $x_p - x \in X_{\varepsilon}$ for $p > N$. But $X_{\varepsilon}$ is a vector space; thus, $x \in X_{\varepsilon}$. 
By (3), we have for arbitrary \( \varepsilon > 0 \),

\[
\zeta \left( \frac{x_p - x}{u_\varepsilon} \right) \leq u_\varepsilon
\]

for \( p > N \). Thus, \( |x_p - x|_\varepsilon < u_\varepsilon < a\Psi(\varepsilon) \) for \( p > N \), and we get \( |x_p - x|_\varepsilon \to 0 \) as \( p \to \infty \). This proves the completeness of the space \( X_\varepsilon \).

**Theorem 6.** Let the function \( \Psi \) and the sequence \( \varphi \) satisfy the assumptions of Theorems 1 and 5. The \( \hat{X}(\Psi) \cap \hat{X}_c \) is a Fréchet space with respect to the \( F \)-norm \( |.|_\varepsilon \).

**Proof.** It is sufficient to show that \( \hat{X}(\Psi) \cap \hat{X}_c \) is a closed subspace of \( \hat{X}_c \) with respect to the \( F \)-norm \( |.|_\varepsilon \). Let \( \tilde{x}_p \in \hat{X}(\Psi) \cap \hat{X}_c \), \( \tilde{x}_p \to \tilde{x} \) in \( \hat{X}_c \). Let \( x_p \in \tilde{x}_p \), \( x \in \tilde{x} \). By the assumption, we have for every \( \lambda > 0 \)

\[
a_\varepsilon \Psi(\omega_\varphi(\lambda(x - x_p), r)) \to 0 \quad \text{as} \quad p \to \infty
\]

uniformly with respect to \( r \). By a property of \( \omega_\varphi \), and the condition \((\Delta_2)\) for \( \varphi \), we have

\[
\omega_\varphi(\lambda z, r) \leq \omega_\varphi(2\lambda(z - x_p), r) + \omega_\varphi(2\lambda z_p, r) \leq K[\omega_\varphi(\lambda(z - x), r) + \omega_\varphi(\lambda z_p, r)].
\]

By properties of \( \Psi \) we have that there exist \( M > 0 \), \( \delta > 0 \) such that for every \( u \) satisfying the condition \( 0 < \Psi(u) \leq \delta \) there holds the inequality \( u \leq M \). Taking \( \lambda > 0 \) fixed we may find a \( p_1 \) such that \( \Psi[\omega_\varphi(\lambda(x - x_p), r)] < \delta \) for \( p \geq p_1 \), and in consequence we obtain that \( \omega_\varphi(\lambda(x - x_p), r) \leq M \) for \( p \geq p_1 \), with an \( M > 0 \). Let \( m \) be such that \( K \leq 2^m \). Applying the inequality \( \Psi(u + v) \leq \Psi(2u) + \Psi(2v) \) and condition \((\Delta_2)\) for small \( u \) with a constant \( K_1 > 0 \), we thus obtain

\[
\Psi(\omega_\varphi(\lambda z, r)) \leq \Psi[2K\omega_\varphi(\lambda(z - x_p), r)] + \Psi[2K\omega_\varphi(\lambda z_p, r)] \leq K_1^{m+1}[\Psi(\omega_\varphi(\lambda(z - x_p), r)) + \Psi(\omega_\varphi(\lambda z_p, r))]
\]
for \( p \geq p_1 \). Let us choose an arbitrary \( \varepsilon > 0 \). Then there exists a \( p_0 \geq p_1 \) such that

\[
a_r \varPsi(\omega_\varphi(\lambda(x - x_{p_0}), r)) < \frac{\varepsilon}{2} K^{-m-1}_1.
\]

But \( x_{p_0} \in X(\Psi) \) and so, by Theorem 1, we have

\[
a_r \varPsi(\omega_\varphi(\lambda x_{p_0}, r)) \to 0 \quad \text{as } r \to \infty.
\]

Hence there exists an \( r_0 \) such that

\[
a_r \varPsi(\omega(\lambda x_{p_0}, r)) < \frac{\varepsilon}{2} K^{-m-1}_1 \quad \text{for } r \geq r_0.
\]

Consequently,

\[
a_r \varPsi(\omega(\lambda x, r)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \text{for } r \geq r_0.
\]

This shows that \( x \in X(\Psi) \). By Theorem 5, \( x \in X_c \). Hence \( x \in X(\Psi) \cap X_c \), and so \( \tilde{x} \in X(\Psi) \cap \tilde{X}_c \).

Let us remark that Theorems 5 and 6 may be expressed also in a form replacing \( F \)-norm convergence by means of modular convergence with respect to the modular \( \tilde{\varepsilon}(\tilde{x}) = \inf\{\varepsilon(y) : y \in \tilde{x}\} \).

Let us recall that a sequence \((\tilde{x}_n)\) of elements of \( \tilde{X}_c \) is said to be \( \tilde{\varepsilon} \)-Cauchy, if there exists a \( k > 0 \) such that for every \( \varepsilon > 0 \) there is an \( N \) such that \( \tilde{\varepsilon}(k(\tilde{x}_p - \tilde{x}_q)) < \varepsilon \) for all \( p, q > N \). The space \( \tilde{X}_c \) is called \( \tilde{\varepsilon} \)-complete, if any \( \tilde{\varepsilon} \)-Cauchy sequence is \( \tilde{\varepsilon} \)-convergent to an element \( \tilde{x} \in \tilde{X}_c \).

There hold the following theorems, proofs of which are analogous to those of Theorems 5 and 6:

**Theorem 7.** Under the assumptions of Theorem 5, the space \( \tilde{X}_c \) is \( \tilde{\varepsilon} \)-complete.

**Theorem 8.** Under the assumptions of Theorem 6, the space \( \tilde{X}(\psi) \cap \tilde{X}_c \) is \( \tilde{\varepsilon} \)-complete.

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References


