ABSTRACT. Several properties of weakly $p$-summable sequences and of the scale of $p$-converging operators (i.e., operators transforming weakly $p$-summable sequences into convergent sequences) in projective and natural tensor products with an $l_p$ space are considered. The last section studies the Dunford-Pettis property of order $p$ (i.e., every weakly compact operator is $p$-convergent) in those spaces.

0. INTRODUCTION

In this paper several properties of the scale of $p$-converging operators in projective and natural tensor products with an $l_p$ space are considered. This scale, introduced in [2] and [3], is intermediate between the ideals of unconditionally converging operators and the ideal of completely continuous or Dunford-Pettis operators. Since $p$-converging operators are characterized by the property of sending weakly-$p$-summable sequences into convergent ones, a part of the study is devoted to a special class of subsets of vector sequence spaces, termed almost compact sets, nontrivial examples of which are, in certain spaces, precisely the weakly-$p$-summable sequences, $1 \leq p < +\infty$. Section 2 characterizes the compact sets of $l_p \tilde{\otimes}_\sigma X$ and $l_p \tilde{\otimes}_\sigma^p X$, extending results of Leonard [8] and Bombal [1]. Section 3 considers the Dunford-Pettis properties of order $p$ in projective and natural tensor product spaces of $l_p$ and a Banach space $X$. For $l_p$-sums of sequences of Banach spaces, generalizations of results of Bombal [1] are obtained. Those properties were introduced in [3].

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1. BACKGROUND

Throughout the paper $p^*$ denotes the conjugate number of $p$. We base our approach to the properties of natural and projective tensor products on the use of the representations of those spaces as sequence spaces. A sequence $(x_n)$ in a Banach space $X$ is said to be weakly-$p$-summable ($p \geq 1$) if for every $x^* \in X^*$ the sequence $(x^*(x_n))$ is in $l_p$; equivalently (see [7] 19.4), if there is a constant $C > 0$ such that, for each $(\xi_n)$ in $l_p$, $w_p((x_n)) = \sup_k \left\{ \|\sum_{n=1}^k \xi_n x_n\| : \|\xi_n\|_{l_p} \leq 1 \right\} \leq C < +\infty$. (Here, if $p = 1$, $c_0$ plays the role of $l_\infty$.) It is said to be absolutely-$p$-summable, when $p = 1$, if $s_p((x_n)) = \left( \sum_{n=1}^\infty \|x_n\|^p \right)^{1/p} < +\infty$. (If $p = +\infty$, the $l_p$ norm has to be replaced by the sup norm.) It is said to be strongly-$p$-summable for $p \geq 1$ if $\sigma_p((x_n)) = \sup \{ \|\sum_{n=1}^\infty f_n(x_n)\| : w_p((f_n)) \leq 1, (f_n) \in X^* \} < +\infty$. Following [7], we shall denote by $l_p[X]$, $l_p\{X\}$ and $l_p<X>$ respectively the spaces of weakly-$p$-summable, absolutely-$p$-summable and strongly-$p$-summable sequences of $X$, endowed with their natural topologies: those induced by the norms $w_p$, $s_p$ and $\sigma_p$, respectively. The following isometries are well-known (see [7] 19.4.3): $l_p[X] = L(l_p, X)$, for $1 < p < +\infty$, and $l_1[X] = L(c_0, X)$. The symbols $\pi$ and $\varepsilon$ shall denote the projective and injective norms on the space $l_p \otimes X$: they are, respectively, the strongest and coarsest crossnorms (i.e. norms satisfying $\|x \otimes y\| = \|x\| \|y\|$) which is possible to define on that space. The symbol $\Delta_p$ denotes the norm induced by $s_p$ over $l_p \otimes X$; the topology induced by $s_p$ is termed the natural topology. We shall denote by $l_p \hat{\otimes}_\pi X$, $l_p \hat{\otimes}_\sigma X$ and $l_p \hat{\otimes}_\Delta X = l_p\{X\}$ the completion of $l_p \otimes X$ with respect to $\varepsilon$, $\pi$, and $\Delta_p$, respectively. The space $l_p \hat{\otimes}_\sigma X$ also admits a representation as a vector sequence space: it is the closed subspace of the space $l_p<X>$ formed by those sequences which are the limit of their finite sections; this can be deduced without difficulty from [5], where it is proved that the norm $\sigma_p$ induces $\pi$ over $l_p \otimes X$.

Let $E$ be any of the spaces $l_p \hat{\otimes}_\sigma X$ or $l_p \hat{\otimes}_\Delta X$, $p_k$ be the continuous projection onto the $k$th coordinate, and $i_k$ the canonical inclusion of $X$ into the $k$th coordinate. If $T:E \rightarrow Y$ is a continuous operator, then a sequence of operators $T_k \in L(X, Y)$ exists such that $T = \sum_k T_k p_k$: explicitly, $T_k = T i_k$. We shall say that $(T_k)$ is the representing sequence of $T$. If $(X_k)$ is a sequence of Banach spaces, and $T$ is an operator from the Banach space $(\sum_k X_k) = \{ x = (x_k) \in \Pi X_k : \|x\|_p = (\sum_k \|x_k\|_p)^{1/p} < +\infty \}$ into $Y$, then the sequence $(T_k)$ defined by $T_k = T i_k$ is again called the representing sequence of $T$ (cf. [1]).
We shall consider the following operator ideals: The ideal $L$ of all continuous operators; the ideal $W$ of weakly compact operators; the ideal $U$ of unconditionally converging operators, i.e., those sending weakly-1-summable sequences into unconditionally summable sequences; the ideal $K$ of compact operators; and the ideal $DP$ of completely continuous or Dunford-Pettis operators, i.e., those sending weakly convergent sequences into convergent ones.

**Definition.** We say that an operator $T\in L(X,Y)$ is $p$-converging, $1\leq p < +\infty$, if it transforms weakly-$p$-summable sequences of $X$ into norm null sequences of $Y$. We shall use $C_p$ to denote the ideal of $p$-converging operators.

The classes $C_p$ form injective, non-surjective closed operator ideals. It is not difficult to see that $C_1 = U$ and, with the convention that the weakly-$\infty$-summable are the weakly null sequences, that $C_\infty = DP$. A characterization of $p$-converging operators is contained in the following proposition (see [3]):

**Proposition 0.** Let $X$ be a Banach space, and $1 \leq p < +\infty$. If $p > 1$ the operator $Id(X)$ belongs to $C_p$ if and only if all operators from $l_p$ into $X$ are compact. If $p = 1$, $Id(X)$ belongs to $C_1$ if and only if all operators from $c_0$ into $X$ are compact.

### 2. COMPACT SETS

We shall study in Section 3 the relation between the membership of an operator $T$ in a class $C_p$, and the membership of the operators forming its representing sequence in that same class. To this end, we shall introduce a class of subsets which have something of the flavour of compact sets.

**Lemma 1.** Let $1 \leq p < +\infty$. Let $X$ and $Y$ be Banach spaces. Consider a set $A \subset l_p \hat{\otimes}_\pi X$ (resp. $A \subset l_p \hat{\otimes}_\sigma X$). The following are equivalent:

1. For each continuous operator $T \in L(l_p \hat{\otimes}_\pi X, Y)$ (resp. $T \in L(l_p \hat{\otimes}_\sigma X, Y)$), the representing sequence of $T$ converges to $T$ uniformly over $A$. 

2. \( \lim_{N \to +\infty} \sup_{x \in A} \| (x_n)_{k=1}^{\infty} \|_r = 0 \) (resp., \( \lim_{N \to +\infty} \sup_{x \in A} \| (x_k)_{i=1}^{\infty} \| = 0 \)).

Proof. That \( 1 \Rightarrow 2 \) is obvious. Let us show that \( 2 \Rightarrow 1 \) for the case of the projective tensor product. Let \( (x_n) \) be any sequence in \( A \). Then

\[
\left\| T(x_n) - (T_1(x_1), T_2(x_2), \ldots, T_N(x_N), 0, 0, \ldots) \right\|_r =
\]

\[
= \| T(0, 0, \ldots, 0, x_{N+1}, x_{N+2}, \ldots) \|_r \leq \| T \| \pi([0, 0, \ldots, 0, x_{N+1}, x_{N+2}, \ldots])
\]

and this converges uniformly on \( A \) by Condition 2.

The computations for the natural product are very similar. \( \blacksquare \)

Definition. Let \( p < +\infty \). A set \( A \subset l_p \otimes X \) (resp. \( A \subset l_p \otimes X \)), is said to be almost-compact if it satisfies either of the equivalent conditions of Lemma 1.

Proposition 2. Let \( p < +\infty \). A subset \( A \subset l_p \otimes X \) (resp. \( A \subset l_p \otimes X \)), is relatively compact if and only if it is bounded, almost-compact, and its continuous projections \( p_k(A) \) are relatively compact in \( X \) for all \( k \in \mathbb{N} \).

Proof. It is easy to see that all the conditions are necessary. They are also sufficient: Let \( (x^*) \) be a sequence contained in \( A \subset l_p \otimes X \). Condition 1 of Lemma 1 and a diagonal argument show that a certain sub-sequence, again denoted \( (x^*) \), exists having pointwise convergence to an element \( x \). To verify that the convergence occurs in the projective norm, it is only necessary to take, in the following expression, the supremum over all elements \( x^* \) in the unit ball of \( l_p[X^*] \):

\[
\sum_{k=1}^{N} |<x_k^*, x_k^*>| \leq \sum_{k=1}^{N} |<x_k^*, x_k^*>| +
\]

\[
+ \sum_{k=N+1}^{\infty} |<x_k^*, x_k^*>| + \sum_{k=N+1}^{\infty} |<x_k, x_k^*>|,
\]

and observe that the first summand can be made, for large \( N \), less than \( \epsilon \); since \( A \) is almost compact, the second and third summands tend to zero when \( N \) tends to infinity.
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The proof for the natural product is analogous.

Remark. Lemma 1 and Proposition 2 have been proved in [1] and [8] for \( l_p(X) \). The referee has informed us that this proposition is a particular case of an old theorem due to Mazur, who stated it for the case of a Banach space having a basis, and that a more general result has been established by Goes and Welland as follows:

**Theorem ([6] Thm. 1.)** Let \( X \) be a complete locally convex topological vector space. Let \( A \) be a bounded subset of \( X \) and \( \{P_\beta\}_{\beta \in I} \) a net in \( L(X, X) \). Then \( A \) is relatively compact if \( \{P_\beta\}_{\beta \in I} \) converges uniformly to the identity on \( A \) and \( P_\beta(A) \) is relatively compact for each \( \beta \in I \).

Proposition 2 follows taking \( P_N((x_1, n, x_N, 0, 0, \ldots)) \) for \( N \in \mathbb{N} \). We have left the proof of Proposition 2 for the sake of completeness.

Nontrivial examples of almost compact sets in natural and projective tensor products are provided by the next proposition.

**Proposition 3.** Assume that \( X \) is a Banach space and that \( 1 \leq p, r < \infty \). If \( r < p^* \), then a weakly-\( r \)-summable sequence of \( l_p \otimes_r X \) or \( l_p \otimes_{a_r} X \) is an almost compact set. For \( p = 1 \), a weakly null sequence of \( l_1(X) \) is an almost compact set.

**Proof.** We first show the proof for the projective product. Let \( (a^n) \) be a weakly-\( r \)-summable sequence in \( l_p \otimes_r X \). Assume that \( A = \{a^n : n \in \mathbb{N}\} \) is not an almost compact set. In that case, an \( \varepsilon > 0 \) and two sequences \( (n_i) \) and \( (N_i) \) of naturals exist such that if \( I_i \) denotes the set \( \{N_i + 1, \ldots, N_i + 1\} \) and \( P_i : l_p \otimes_r X \rightarrow l_p \otimes_r X \) denotes the projection over the indices of \( I_i \), then

\[
\pi_p(P_i(a^n)) > \varepsilon.
\]

Elements \( z_i \in (l_p \otimes_r X)^* = L(l_p, X^*) \) with \( \|z_i\| \leq 1 \) can be chosen such that \( \langle P_i(a^n), z_i \rangle \rangle > \varepsilon \). The proof of [4, Thm. 1] shows that if \( Q_i : l_p \rightarrow l_p \) denotes the projection over the indices of \( I_i \), then \( \langle P_i(a^n), z_i Q_i \rangle \rangle > \varepsilon \).

Once more, the proof of [4, Thm. 1] shows that the operator \( B : l_p \otimes_r X \rightarrow l_p \).
defined by \( B(Y) = \langle P, y, z, Q, \rangle \) is continuous. By [3, Prop. 1.6.], it transforms \((a^n)\) into a norm-null sequence of \( l_p\), which is a contradiction.

The proof for the natural product is essentially the same. We shall give it for the sake of completeness: If \( A = \{a^n; n \in \mathbb{N}\} \) is not almost compact, then an \( \varepsilon > 0 \) and two sequences \((n_i)\) and \((i_i)\) of naturals exist such that

\[
\sum_{k=N_i}^{k=N_{i+1}} \|a_k^n\|^p > \varepsilon.
\]

Normalized elements \( x^* (k) \in X^* \) can be chosen such that:

\[
\langle x^* (k), a_k^n \rangle = \|a_k^n\|. \quad \text{If } y^*_k = x^* (k), \text{ for } N_i \leq k < N_{i+1}, \text{ then } (y^*_k) \text{ is a bounded sequence of } X^* \text{ which defines an element of } L(l_p (X), l_p). \text{ This operator transforms } (a^n) \text{ into a weakly-}\( r \)-summable sequence of } l_p, \text{ which must be norm-null (see [3, Prop. 1.6.]). Thus one has}
\[
\lim_{N \to +\infty} \sup_{n \in \mathbb{N}} \left( \sum_{k=N}^{k=+\infty} \langle y^*_k, a_k^n \rangle \right)^{\frac{1}{p}} = 0,
\]

which is a contradiction.

The proof for the case \( p = 1 \) follows closely that of the natural product, and it is only necessary to recall that \( l_1 \) has the Schur property, i.e.: weakly null sequences are norm null. That yields the proof for the projective tensor product since \( l_1 \bigotimes_{\pi} X = l_1 (X) \). In other words: the statement holds for \( p = 1 \) and \( r = \infty \).

**Remark.** Let \( X_n \) be a sequence of Banach spaces, and \( 1 \leq p < +\infty \). A set \( A \subseteq (\bigoplus_{n} X_n)_p \) is said to be almost compact if Conditions 1 or 2 of Lemma 1, with suitable modifications, are satisfied. In this form, Propositions 2 and 3 can be translated to \( l_p \)-sums of sequences of Banach spaces.

### 3. DUNFORD-PETTIS PROPERTIES

A Banach space \( X \) is said to have the Dunford-Pettis property (DPP) if weakly compact operators defined on \( X \) are completely continuous, that is, if for any Banach space \( Y: W(X, Y) \subseteq DP(X, Y) \). Typical examples of Banach spaces having DPP are \( L_\infty \) and \( L_1 \) spaces. No reflexive Banach space can have DPP. Weakened versions of the Dunford-Pettis property were in-
introduced in [3]. A Banach space $X$ is said to have the Dunford-Pettis property of order $p \geq 1$, if $\mathcal{W}(X, Y) \subseteq C_p(X, Y)$ for all Banach spaces $Y$. We shall call this property $DPP_p$. Notice that $DPP_\infty = DPP$. Every Banach space has $DPP_1$. Other examples are (see [3] for details): $l_p$ has $DPP$, for all $r < p^*$; $L_0[0, 1]$ has $DPP$, for $r < \min\{p^*, 2\}$; Tsirelson's space has $DPP_r$ for all $r < +\infty$, but not $DPP$ since it is reflexive; if $id(X) \in C_p$, then $C(K, X)$ has $DPP_p$.

**Lemma 4.** Let $1 \leq p < +\infty$. Let $(X_n)$ be a sequence of Banach spaces. Assume that $E$ represents any of the spaces $(\sum \mathcal{X}_n)_p$ or $l_p \bigotimes_n X$, and that $T$ is a continuous operator from $E$ into a Banach space $Y$, having $(T_k)$ as a representing sequence. If $r < p^*$ (or $p = 1$ and $r = \infty$), then $T$ is $r$-converging if and only if each $T_k$ is $r$-converging.

**Proof.** Let $(a^*_n)$ be a weakly-$r$-summable sequence of $E$. Since $(a^*_n)$ is an almost compact set, the convergence of $(T_k)$ to $T$ is uniform over the set $(a^*_n)$. Furthermore, $T_k((a^*_n))$ is relatively compact in $Y$ since $T_k$ is $r$-converging. The relationship

$$T((a^*_n)) \subseteq \sum_{k=1}^{k=N(n)} T_k((a^*_n)) + \varepsilon B_Y$$

implies that $T((a^*_n))$ is relatively compact, and therefore $(T a^*_n)$ must be norm-null.

**Remark.** When $p^* \leq r < \infty$ the result is clearly false: simply consider the example $l_p(l_1)$ and $T = id$.

**Proposition 5.** Let $A$ denote an operator ideal and $r < p^*$ (or $p = 1$ and $r = \infty$). With the same notation as in Lemma 4, $A((\sum \mathcal{X}_n)_p, Y) \subseteq C_r((\sum \mathcal{X}_n)_p, Y)$ if and only if, for all $n$, $A(X_n, Y) \subseteq C_r(X_n, Y)$. Moreover $A(l_p \bigotimes_n X) \subseteq C_r(l_p \bigotimes_n X)$ if and only if $A(X, Y) \subseteq C_r(X, Y)$.

**Remark.** Recalling that $C_1 = U$ and that $C_\infty = DP$, one sees that these results include and generalize the following results of Bombal [1]: Theorem 1.5, part a) for the unconditionally converging operators ($p = 1$,
Theorem 6. Let \( 1 \leq p < + \infty \). Assume that \( r < p^* \) (or \( p = 1 \) and \( r = \infty \)): these are the cases when \( l_p \) has DPP. Assume that \( X \) also has DPP. Then \( l_p \hat{\otimes} A X \) and \( l_p \hat{\otimes} \Delta X \) also have DPP.

Proof. Let \( E \) denote any of those spaces, and let \( T : E \to Y \) be a weakly compact operator. Since \( X \) has DPP, the operators \( (T_n) \) in the representing sequence of \( T \), which necessarily are weakly compact, are \( p \)-converging. By Lemma 4, \( T \) must also be \( p \)-converging.

References