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Multiplicity of a Foliation on Projective Spaces along an Integral Curve

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ABSTRACT. We compute the global multiplicity of a 1-dimensional foliation along an integral curve in projective spaces. We give a bound in the way of Poincaré problem for complete intersection curves. In the projective plane, this bound give us a bound of the degree of non irreductible integral curves in function of the degree of the foliation.

0. INTRODUCTION

Let $\mathscr{F}$ be a foliation by lines in the complex projective space $\mathbb{P}_n$. Let us take homogeneous coordinates $X_0, \ldots, X_n$. There exists an homogeneous vector field

$$D=\sum_{i=0}^{n} A_i\,(X_0, \ldots, X_n)\,\frac{\partial}{\partial X_i}, \quad deg(A_i)=d, \quad g.c.d.\{A_i\}=1$$

such that $\mathscr{F}$ is given by any element of the set of vector fields $\mathscr{D}=\{D+H.R.\}$ where $R=\sum_{i=0}^{n} X_i\,\frac{\partial}{\partial X_i}$ and $H=H(X_0, \ldots, X_n)$ is an homogeneous polynomial with $deg(H)=d-1$, [1], [6]. This number d is said to be *the degree* of $\mathscr{F}$, $deg(\mathscr{F})$. The solutions of the equations $\frac{A_0}{X_0}= \ldots =\frac{A_n}{X_n}$ form the set of *singularities* of $\mathscr{F}$ and we assume that this set $Sing(\mathscr{F})\subset\mathbb{P}_n$ is finite.

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Let $S \subset \mathbb{P}_n$ be a subvariety of $\mathbb{P}_n$ and $I \subset \mathbb{C}[X_0, \ldots, X_n]$ be the homogeneous ideal defining S. We say that S is a *solution* of $\mathcal{F}$ if $D(I) \subset I$. In particular, S is an *integral curve* if $dim(S) = 1$.

Let $x_1 = \dfrac{X_1}{X_0}, \ldots, x_n = \dfrac{X_n}{X_0}$ be a system of affine coordinates. The polynomial vector field

$$E = \sum_{i=1}^{n} [A_i(1, x_1, \ldots, x_n) - x_i A_0(1, x_1, \ldots, x_n)] \frac{\partial}{\partial x_i}$$

represents the foliation $\mathcal{F}$ in the affine chart $(X_0 \neq 0)$, but it is independent of H. Let p be the degree of the vector field E, i.e., the maximum of the degrees of the coefficients. It is easy to see that $p = d$ if the hyperplane at infinity is a solution of $\mathcal{F}$ and $p = d + 1$ in the other cases.

In [10], Poincaré study the problem of finding a bound of the degree of an algebraic integral curve in function of the degree of the algebraic differential equation. In general, this problem has no solution. For instance, let us take the equation of degree 1 given by the vector field $E = x \dfrac{\partial}{\partial x} + my \dfrac{\partial}{\partial y}$ and the integral curve $y - x^m = 0$.

The problem of Poincaré has been recently treated by D. Cerveau and A. Lins [3]. They suggest to use the total *multiplicity of a foliation* $\mathcal{F}$ along a projective integral curve C, (see(1.1))

$$m(\mathcal{F}, C) = \sum_{P \in C} m_P(\mathcal{F}, C)$$

and they give some bounds with restrictive conditions on the curve. The main tool used by them to globalize the local multiplicity is the following formula

$$m(\mathcal{F}, C) = 2 - 2g + m(d-1)$$

where m and g are the degree and the genus of the irreducible curve C. This idea is the origin of these notes. In [4], we analyse some properties of $m(\mathcal{F}, C)$. M. Carnicer [2] use this formula for proving that $m \leq d + 2$ when the singularities of the foliation are non dicritical. Namely, let us take a local representation $E = a \dfrac{\partial}{\partial x} + b \dfrac{\partial}{\partial y}$ of $\mathcal{F}$ at a singularity with multiplicity $\nu \geqslant 1$. Let a_ν, b_ν be the components of degree ν of a, b respectively. The singularity is said to be *non dicritical* if $xb_\nu - ya_\nu \not\equiv 0$. Equivalently, the exceptional divisor obtained after blowing-up the singularity is a leaf of the strict transform foliation.

The purpose of this paper is to extend these ideas. In fact, we proof that the above formula is true in the general case, i.e., for non irreducible curves in $\mathbb{P}_n$ (Theorem (1.3)). This allow us to give a result in the way of the Poincaré problem when the curve C is a complete intersection of $n-1$ hypersurfaces (Theorem (1.6)). The second part is devoted to complete some bounds in dimension two (Theorem (2.3) and Proposition (2.6)). In the proofs, it is essential to use formulas to compute the arithmetic genus of the non irreducible plane curves or curves in $\mathbb{P}_n$.

1.1. Let X be a n-dimensional complex manifold and $\mathcal{F}$ be an analytic foliation by lines on X. At every point $P \in X$, the foliation is generated by a (germ of) vector field $E \in Der(O_{X,P})$. Let C be an analytic branch at P which is integral curve of $\mathcal{F}$, i.e., if $I \subset O_{X,P}$ is the ideal of the branch then $E(I) \subset I$. Let $\bar{O}_{C,P}$ be the integral closure of the local ring $O_{C,P}$ of the branch C in its function field. Then, the derivation E defines a derivation $\bar{E}$ of $\bar{O}_{C,P}$. Let T be a generator of the $\bar{O}_{C,P}$-module $Der(\bar{O}_{C,P})$. Then $\bar{E} = f.T$, with $f \in \bar{O}_{C,P}$. If we write ν_P for the valuation of the ring $\bar{O}_{C,P}$, we shall call $m_P(\mathcal{F}, C) = \nu_P(f)$ *the multiplicity of* $\mathcal{F}$ *along* C. It is easy to see that $m_P(\mathcal{F}, C)$ does not depend of the choice of E and T. Let us observe that $m_P(\mathcal{F}, C) = 0$ iff $\mathcal{F}$ is regular at P. Indeed, let $x_1, \ldots, x_n$ be a local system of coordinates at P and $E = \sum_{i=1}^{n} a_i(x_1, \ldots, x_n)\frac{\partial}{\partial x_i}$. If t is a local parameter of C at P and $x_i = x_i(t)$, $i = 1, \ldots, n$ are the parametric equations, then $a_i(x_1(t), \ldots, x_n(t)) = f(t)\, x_i'(t)$ and $m_P(\mathcal{F}, C) = ord_t\left(\frac{a_i(x_1(t), \ldots, x_n(t))}{x_i'(t)}\right)$. Since $\mathcal{F}$ is regular if some of the $a_i(x_1, \ldots, x_n)$ is an unit, then the observation is obvious.

1.2. Let $X = \mathbb{P}_n$ and fix a foliation by lines $\mathcal{F}$ of degree d. Let C be an irreducible integral curve of degree m and genus g, and $\pi: \bar{C} \to C$ be the normalization of C. Since $Sing(\mathcal{F})$ is a finite set, then $m(\mathcal{F}, C) = \sum_{Q \in \bar{C}} m_Q(\mathcal{F}, C)$ is a well defined integer number, positive or null. We shall state the next theorem for curves C with several components.

Let C_i, $i=1, \ldots, r$ be the irreducible components of C with degrees m_i and genus g_i respectively. We shall write $m=\sum_{i=1}^{r} m_i$ for the degree of the curve C, $2-2g= \sum_{i=1}^{r} (2-2g_i)$ to define the genus g of C and $m(\mathscr{F}, C)=\sum_{i=1}^{r} m(\mathscr{F}, C_i)$.

1.3. Theorem. *Lef $\mathscr{F}$ be a foliation by lines of degree d on $\mathbb{P}_n$ and C be an (irreducible or not) integral curve of genus g and degree m. Then*

$$m(\mathscr{F}, C)=2 - 2g + m(d - 1).$$

Proof. It is enough to suppose that C is irreducible because for several irreducible components the equality is the sum of the respectives equalities.

Let us fix the homogeneus coordinates $X_0, \ldots, X_n$ such that the hyperplane $(X_0=0)$ has no singular points of $\mathscr{F}$ and the intersection with the curve C consist of m different points. Let $x_i=X_i/X_0$, $i=1, \ldots, n$ be the affine coordinates and let

$$E=\sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i}, \qquad a_i \in \mathbb{C}[x_1, \ldots, x_n], \qquad gcd(a_i)=1$$

be the expression of a generator of $\mathscr{F}$ in the affine open $(X_0\neq 0)$. To compute the degree of E we need a lemma.

1.4. Lemma. *Let $p\geq 2$ be the degree of a polynomial vector field $E=\sum_{i=1}^{n} a_i\frac{\partial}{\partial x_i}$. If the hyperplane at infinity $(X_0=0)$ is a solution of the foliation then E has a pole of order $p-2$ at $(X_0=0)$. In the other cases, E has a pole of order $p-1$.*

Proof of the lemma. It is enough to compute the expression of E for another affine chart, for instance $(X_n\neq 0)$.

Let $y_0=\frac{1}{x_n}$, $y_1=\frac{x_1}{x_n}$, $\ldots$, $y_{n-1}=\frac{x_{n-1}}{x_n}$ be the system of affine coordinates in this new chart. We have

$$E = -y_0^2 a_n \frac{\partial}{\partial y_0} - y_0 \sum_{i=1}^{n-1} (a_i - y_i a_n) \frac{\partial}{\partial y_i} =$$

$$= -\frac{1}{y_0^{p-1}} \left[y_0 b_n \frac{\partial}{\partial y_0} + \sum_{i=1}^{n-1} (b_i - y_i b_n) \frac{\partial}{\partial y_i} \right]$$

where $b_i = y_0^p a_i \left(\frac{y_1}{y_0}, \ldots, \frac{y_{n-1}}{y_0}, \frac{1}{y_0} \right)$.

If $(y_0 = 0)$ is a solution, then y_0 is a factor of the coefficient of $\frac{\partial}{\partial y_0}$ and it can not be factor of the other coefficients. Hence E must have a pole of order $p-1$. Otherwise, y_0 is a factor of all the coefficients and the order of the pole is $p-2$.

Now we proof the theorem. Let us recall that the degree of E is d (resp. $d+1$) if the hyperplane at the infinite is (resp. is not) a solution of $\mathcal{F}$. From the lemma, it follows that every affine generator of $\mathcal{F}$ has a pole or order $d-1$ at the infinite, independently of the system of coordinates. The restriction of E to C give us a vector field $\bar{E}$ which is a derivation of the function field $K(C)$ of the curve. If T is a generator of the $K(C)$-vectorial space of the derivations, one has $\bar{E} = f.T$ with $f \in K(C)$. Let us write v_Q for the valuation of $K(C)$ at the point $Q \in \bar{C}$. Then, the integer $e_Q(E) = v_Q(f)$ depends of E but not of f. By definition, this number coincides with $m_Q(\mathcal{F}, C)$ for $Q \in (X_0 \neq 0)$. Nevertheless, if $Q \in C \cap (X_0 = 0)$ we have $m_Q(\mathcal{F}, C) = 0$ since Q is regular for $\mathcal{F}$, but $e_Q(E) = -(d-1)$ by the above discussion. This implies that

$$\sum_{Q \in \bar{C}} e_Q(E) = m(\mathcal{F}, C) - m(d-1).$$

To finish the proof, it is enough to see that $\sum_{Q \in \bar{C}} e_Q(E) = 2 - 2g$. Let θ be the differential form of $\bar{C}$ dual of the derivation $\bar{E}$. If t is a local parameter of C at $Q \in \bar{C}$ and $\bar{E} = f \frac{d}{dt}$, $f \in K(C)$, one has $\theta = \frac{1}{f} \cdot dt$. It follows that the divisor $div(\theta) = \sum_{Q \in \bar{C}} e_Q(\theta) \cdot Q$ such that $e_Q(\theta) = v_Q(1/f)$ and the divisor $div(E) = \sum_{Q \in \bar{C}} e_Q(E) \cdot Q$ defined over $\bar{C}$ are opposite. But $div(\theta)$ is a canonical divisor and it is well known that the degree is $2g-2$ ([7], ch4, (1.3.3)). Q.E.D.

1.5. Remark. The statement of the theorem (1.4) is of a global character. In the proof there is an interesting local argument. Actually we proof a relation between (Weil) divisors of C. Let $M(\mathcal{F}, C)=\sum_{Q\in C} m_Q(\mathcal{F}, C)\cdot Q$. Let us fix homogeneous coordinates $X_0, \ldots, X_n$ like in the proof and let $H=\sum_{Q\in(X_0=0)} I_Q(C, (X_0=0))\cdot Q$, i.e., the intersection of C and the hyperplane $(X_0=0)$. From the proof of the theorem it follows that

$$M(\mathcal{F}, C)=div(E) + (d-1)H.$$

We say that a curve is a *nodal curve* if its singularities are simple nodes, i.e., they are of normal crossing type.

1.6. Theorem. *Let $\mathcal{F}$ be a foliation by lines in $\mathbb{P}_n$ of degree d and C be an irreducible curve complete intersection of $n-1$ hypersurfaces of degrees $a_1, \ldots, a_{n-1}$. Let us suppose that C sastisfy one of the following conditions:*

a) C is a non singular curve.
b) C is a nodal curve.
Then, one has

$$\sum_{i=1}^{n-1} a_i \leq d+n$$

Furthermore, in the case a) one has the equality iff $\mathcal{F}$ has no singularities over C.

Proof. The main tool for proving this theorem is the use of the following formula to compute the arithmetic genus $\pi=H^1(O_C,C)$ of C ([11], n.78)

$$\pi=\frac{1}{2}ma + 1, \qquad a=-1-n+\sum_{i=1}^{n} a_i,$$

where $m=a_1\cdot\ldots\cdot a_{n-1}$ is the degree of C. It is well known that $g=\pi-\delta$ where

$$\delta = \sum_{P \in Sing(C)} \sum_{Q \geq P} \frac{e_Q(e_Q - 1)}{2}$$

$Q \geq P$ means that Q is an infinitely near point of P and e_Q is the multiplicity of the strict transform of C at Q. From (1.3) we have

$$m(\mathcal{F}, C) - 2\delta = m(d-1-a).$$

Now, let C be a non singular curve. This implies that $\delta = 0$. Since $m(\mathcal{F}, C) \geqslant 0$, we must have $d-1-a \geqslant 0$ and it follows that $\sum_{i=1}^{n-1} a_i \leqslant d+n$.

In the case b), if C is a nodal curve, then δ is the number of nodes of C and $\mathcal{F}$ has a singularity in every one because C is an integral curve. For each branch $Q \in \tilde{C}$ of the node one has $m_Q(\mathcal{F}, C) > 0$. This implies that the contribution of $m(\mathcal{F}, C)$ in every node is two at least. Then, we have $m(\mathcal{F}, C) - 2\delta \leqslant 0$ and it follows that $\sum_{i=1}^{n-1} a_i \leqslant d+n$. Q.E.D.

2.1. Let us consider from now on, the case $n=2$. It is possible to extend the theorem (1.5) to the non irreducible curves. The main argument is to check that the genus formula

$$g = \frac{(m-1)(m-2)}{2} - 2\delta$$

is available in this case. Let C_i, $i = 1, \ldots, r$ be the irreducible components of C and m_i, g_i, δ_i be the respective multiplicities, genus and δ-values. Let us remark that δ is not equal to $\sum_{i=1}^{r} \delta_i$ in general. In fact, one has

2.2. Lemma. *If C is a non irreducible plane curve then*

$$2 - 2g = m(3-m) + 2\delta.$$

Proof. Since C_i is irreducible, the genus formula give us

$$2-2g_i=m_i(3-m_i) + 2\delta_i.$$

Since $\sum_{i=1}^{r}(2-2g_i)=2-2g$ by definition, it follows that

$$2-2g=m(3-m) + 2\sum_{i=1}^{r}\left(\delta_i + \sum_{i<j} m_i m_j\right).$$

Then, it is enough to see that

$$\delta=\sum_{i=1}^{r}\left(\delta_i + \sum_{i<j} m_i m_j\right).$$

For every point $P\in Sing(C)$ and $Q\geq P$, one has $e_Q=\sum_{i=1}^{r} e_Q(C_i)$ and so

$$e_Q(C)\ (e_Q(C)-1)=\sum_{i=1}^{r} e_Q(C_i)\ (e_Q(C_i)-1) + 2\sum_{i=1}^{r}\left(\sum_{i<j} e_Q(C_i)e_Q(C_j)\right).$$

Let $I_P(C_i, C_j)$ be the intersection multiplicity of C_i and C_j at P. The Noether Formula says that

$$I_P(C_i, C_j)=\sum_{Q\geq P} e_Q(C_i)e_Q(C_j)$$

and Bezout Theorem along $C_i\cap C_j$ implies that

$$m_i\cdot m_j=\sum_{P\in C_i\cap C_j}\ \sum_{Q\geq P} e_Q(C_i)e_Q(C_j).$$

To finish, let us sum all this products for $j<i$. Q.E.D.

2.3. Theorem. *Let $\mathcal{F}$ be a foliation by lines of degree d on $\mathbb{P}_2$ and C be a not irreducible integral curve of degree m. Let us suppose that C satisfy one of the following conditions*

a) C is a non singular curve.
b) C is a nodal curve.

Then $m\leqslant d+2$. Furthermore, in the case a) one has the equality iff $\mathcal{F}$ has no singularities over C.

Proof. Since the genus formula is available in this hypothesis (Lemma(2.2)), the proof is analogous to (1.5). Q.E.D.

2.4. Remark. Let $E=a_1\dfrac{\partial}{\partial x_1}+a_2\dfrac{\partial}{\partial x_2}$ be a local generator of $\mathscr{F}$. Recall that the Milnor number of $\mathscr{F}$ at P is defined as

$$\mu_P(\mathscr{F})=dim_{\mathbb{C}}(O_{\mathbb{P}_2,P}/(a_1,\ a_2)).$$

Let us assume that the integral curve C is regular at P and $\mu_P(\mathscr{F})=1$. This implies that a_1 and a_2 are regular and transversal. Since $\mu_P(\mathscr{F})$ is a local invariant, then we can take an analytic system of coordinates $(x_1,\ x_2)$ such that $(x_1=0)$ is a local equation of the curve C, $a_1=x_1\cdot u_1$ for an unit u_1 and $ord_{x_2}(a_2)=1$. Hence, $m_P(\mathscr{F},\ C)=1$.

Now, let C be a nodal curve like in b) of (2.3). If $\mu_P(\mathscr{F})=1$ for all nodes of C then $m=d+1$ iff the singularities of $\mathscr{F}$ over C are exactly the nodes of C.

2.5. If C is an irreducible non singular curve, the bound $m\leqslant d+2$ of (2.3) (or(1.5)) can be improved by using the "$AF+BG$" Noether's theorem. To proceed, we shall use the dual characterization of the foliation $\mathscr{F}$ by the homogeneous differential 1-form $\Omega=\sum_{i=1}^{3}B_idX_i$ such that $deg(B_i)=d+1$ and $\Omega=D\wedge R$ where $R=\sum_{i=1}^{3}X_i\dfrac{\partial}{\partial X_i}$ ([1] (4.2), [**8**] (I.1)). Let $F(X_1,\ X_2,\ X_3)=0$ be the homogeneous equation of the curve C. Since C is an integral curve of $\mathscr{F}$, we have $\Omega\wedge dF=0$. A foliation such that its solutions are the level curves of a rational function F/G^m with $deg(G)=1$ is an *exact differential of* C.

2.6. Proposition. *Let C be a non singular integral curve of a foliation $\mathscr{F}$ with $deg(C)=m$ and $deg(\mathscr{F})=d$. Then $m\leqslant d+1$ and $m=d+1$ iff $\mathscr{F}$ is an exact differential of C.*

Proof. Let $F_i=\partial F/\partial X_i$, $i=1,2$. Since C is an integral curve of $\mathscr{F}$ it follows that $B_2F_1-B_1F_2$ is in the ideal generated by F in every local ring $O_{C,P}$ and $\nu_P(B_2)\geqslant\nu_P(F_2)$. Then, Noether's conditions are satisfied and this implies that there exists homogeneous polynomials L, M such that $B_2=LF+MF_2$ with $deg(L)=d+1-m\geq 0$ and $deg(M)=d+2-m$. Moreover, if $m=d+1$ then M is a line and $\Omega=MdF-mFdM$, i.e., $\mathcal{F}$ is an exact differential of C. Q.E.D.

2.7. Corollary. *Let C be an integral curve of $\mathcal{F}$. Then C contains at least one singularity of $\mathcal{F}$.* ([9]).

Indeed, if P is a singular point of C, then it is singular for $\mathcal{F}$ too. But if C is not singular, from (2.3) and (2.6) it follows the corollary.

The results (2.3) and (2.6) were already proved in [3] by using different arguments.

2.8. Remark. The formula (1.3) allow us to bound the number of integral lines of a plane foliation in very general cases. By ([5], (2.6)), a foliation with an infinity of integral lines is a radial foliation. Let us suppose that $\mathcal{F}$ is a non radial foliation of degree d and let $R_1, \ldots, R_\alpha$ be the integral lines of $\mathcal{F}$ with the condition that at least one of them, for instance R_1, intersects the other ones in $\alpha-1$ different points-this is the case when $R_1, \ldots, R_\alpha$ are in general position. Let us take $C=R_1$ in the theorem (1.3). Since $g=0$, $m=1$, then $m(\mathcal{F}, R_1)=d+1$. But, for the points $P\in R_1\cap R_i$, $i=2, \ldots, \alpha$, one has $m_P(\mathcal{F}, R_1)\geq 1$ and, consequently $\alpha\leq d+2$.

In the hypothesis of M.Carnicer[2], i.e., all the points of $Sing(\mathcal{F})\cap C$ are non dicritical singularities of $\mathcal{F}$, then the number α of integral lines is bounded by $d+2$ too. To see this it suffices to take C equal to the product of the α integral lines.

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