Dunford-Pettis-like Properties of Continuous Vector Function Spaces

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ABSTRACT. In this paper, the structure of some operator ideals $\mathcal{A}$ defined on continuous function spaces is studied. Conditions are considered under which "$Te \mathcal{A}$" and "the representing measure of $T$ takes values in $\mathcal{A}$" are equivalent for the scales of $p$-converging ($C_p$) and weakly-$p$-compact ($W_p$) operators. The scale $C_p$ is intermediate between the ideals $C_p = \mathcal{U}$ (unconditionally summing operators), and $C_{\infty} = \mathcal{B}$ (completely continuous operators), which have been studied by several authors (Bombal, Cembranos, Rodríguez-Salinas, Saab). The dual scale $W_p$ is intermediate between the ideals $\mathcal{K}$ (compact operators) and $W_{\infty} = \mathcal{W}$ (weakly compact operators), and the results presented have a close connection with those of Diestel, Núñez and Seifert.

1. PRELIMINARIES

In this paper, $B(\Sigma, X)$ denotes the space of all bounded $X$-valued $\Sigma$-measurable functions; if $1 \leq p \leq \infty$, $p^*$ denotes the conjugate number of $p$; if $p=1$, $l_p$, plays the role of $c_0$.

1.1. Definition. A sequence $(x_n)$ in a Banach space $X$ is said to be weakly-$p$-summable ($1 \leq p \leq \infty$) if $(x^* x_n) \in l_p$ for all $x^* \in X^*$, or equivalently, if there is a constant $C > 0$ such that
We shall denote by \( w_p((x_n)_n) \) the infimum of those constants \( C \).

We shall say that \( (x_n) \) is weakly-p-convergent to \( x \in X \) if \( (x_n-x) \) is weakly-p-summable. Weakly-\( \infty \)-convergent sequences are simply the weakly convergent sequences.

1.2. Definition. Let \( 1 \leq p \leq \infty \). An operator \( T \in \mathcal{L}(X,Y) \) is said to be \( p \)-convergent if it transforms weakly-p-summable sequences into norm null sequences. We shall denote by \( C_p \) the class of \( p \)-convergent operators.

When \( p=\infty \) this definition gives the ideal \( B \) of completely continuous operators, that is to say, those transforming weakly null sequences into norm null sequences. When \( p=1 \), it is easy to verify that \( C_1=U \), the ideal of unconditionally summing operators, i.e., those transforming weakly-1-summable sequences into summable ones. Obviously \( C_q \subseteq C_p \) when \( p < q \).

The scale of \( C_p \) ideals are intermediate between the ideals \( B \) and \( U \). It is clear (from the definition) that \( C_p \) are injective operator ideals, and, since any separable Banach space is a quotient of \( l_p \), they are not surjective. On the other hand, it is easy to see that \( C_p \) is closed: let \( (T_n) \) be a sequence of \( p \)-converging operators with limit (in the operator norm) \( T \). If \( (x_n) \) is a weakly \( p \)-summable sequence and \( \varepsilon > 0 \), then \( \|Tx_n\|_p \leq \varepsilon \|x_n\| + \|T_0x_n\| \leq 2\varepsilon \) and \( (Tx_n) \) is norm null.

1.3. Definition. A bounded set \( K \) in a Banach space is said to be relatively weakly-p-compact \((1 \leq p \leq \infty)\) if every sequence in \( K \) has a weakly-p-convergent sub-sequence. An operator \( T \in \mathcal{L}(X,Y) \) is said to be weakly-p-compact, \( 1 \leq p \leq \infty \), if \( T(B) \) is relatively weakly-p-compact. We shall denote by \( W_p \) the ideal of weakly-p-compact operators.

The \( W_p \) operators are meant to be a gradations of the class of weakly
compact operators. It is clear that $W_w=W$ (weakly compact operators), and it is easy to see that $id(X)\in W_1$ if and only if $X$ is finite dimensional. Obviously $W_p\subseteq W_q$ when $p<q$.

The ideals $W_p$ are injective and surjective but not closed. The ideal $W_1$ is not closed since $W_1\neq W^2=K$, the ideal of compact operator (see [14]). To see $W_p$ is not closed for $p>1$, we apply [14, Prop. 1.6] to the diagram:

\[
\begin{array}{ccc}
l_p & \rightarrow & l_1 \\
\downarrow & & \downarrow \\
l_p & & l_q
\end{array}
\]

for $1<p<r<q$. The left arrow is the identity and the right arrow is the inclusion, which belongs to $W_q$. If this operator ideal was closed, the middle inclusion should also be in $W_{q^*}$, which is not, since $C_p W_p=K$ and $W_p$.

1.4. Proposition. Let $1<p<\infty$, then $id (l_p)\in W_{p^*}$.

Proof. Let $(x_n)$ be a bounded sequence in $l_p$. It admits a weakly convergent sub-sequence $(x_{n_k})$. Let $x$ be its weak limit, and let us call $y_n=x_n-x$. If $(y_n)$ is norm null, we have finished. If not, and we have $\|y_n\|\geq \varepsilon>0$ for some sub-sequence, applying the Bessaga-Pelczynski selection principle, we obtain a new sub-sequence, equivalent to the canonical basis $(e_a)$ of $l_p$, which is weakly $p^*$-summable.

An easy consequence is:

1.5. Proposition. $\mathcal{U}(l_p,X)=K(l_p,X)$ if and only if $id(X)\in C_p$.

Moreover, an operator $T$ belongs to $C_p(X,Y)$ if and only if for each $j\in \mathcal{U}(l_p,X)$ the composition $T\circ j$ is compact. From this and the proof of (2.5) we obtain

1.6. Proposition. If $T\in W_p(X,Y)$ then $T^*\in C_r(Y^*,X^*)$ for all $r<p^*$.
1.7. Corollary. Let $1 < p < \infty$, $\text{id}(l_p) \subseteq C_r$ for all $r < p^\ast$.

Remarks.

1. The progression expressed by (1.7) suddenly breaks down when $p < 1$, due to [17], where it is shown that a weakly-1-summable sequence $(x_n)$ exists in each $l_p$, $p < 1$, for which $\|x_n\| \to +\infty$.

2. Regarding Proposition 1.5, this result is equivalently to Pitt's lemma: $\mathcal{L}(l_p, l_q) = K(l_p, l_q)$ if and only if $p > q$.

For $L_p$ spaces the situation is:

1.8. Proposition.

a) If $2 \leq p < \infty$ then $\text{id}(L_p) \subseteq W_2$.

b) If $1 < p < 2$ then $\text{id}(L_p) \subseteq W_\rho^\ast$.

Proof. Part a) can be obtained by using the Kadec-Pelczynski alternative: every normalized weakly null sequence in $L_p$ has a subsequence equivalent either to the unit vector basis of $l_p$ or the unit vector basis of $l_2$.

Part b) follows from a standard duality argument. If $(x_n)$ is a normalized weakly null sequence in $L_p$ and $(x_k)$ is a basic sub-sequence of $(x_n)$, consider a bounded sequence $(y_i)$ of biorthogonal functionals in $L_{p^\ast}$, and (again) the Kadec-Pelczynski alternative.

1.9. Examples. (See [21] for details). We shall abbreviate $id(X) \subseteq C_p$ (resp. $id(X) \subseteq W_p$) by saying $X \subseteq C_p$ (resp. $X \subseteq W_p$).

a) If $1 \leq p < \infty$, $l_p \subseteq C_r$ for $1 \leq r < p^\ast$, and $l_p \subseteq W_\rho^\ast$ for $1 < p < \infty$ (see (1.4) and (1.7)).
b) If $1 \leq p < \infty$, $L_p(\mu) \in C_r$ for $r < \min(2, p^*)$. If $1 < p < \infty$, $L_p(\mu) \in W_r$ for $r = \max(2, p^*)$ (see (1.8) and (1.6)).

c) Tsirelson's space $T$ is such that $T \in C_p$ for all $p \neq \infty$ (see [7]).

d) Tsirelson's dual space $T^*$ is such that $T^* \in W_p$ for all $p > 1$ (see [7]).

e) Super-reflexive spaces belong to some class $W_p$ and, consequently, to some class $C_q$ (see [6]).

f) If $X, l_* \in W_p$ then so does $l_*(X)$ (see [8]).

It is well-known [12] that every operator $T$ from $C(K, X)$ to $Y$ has a finitely additive representing measure $m$ of bounded semi-variation, defined on the Borel $\sigma$-field $\Sigma$ of $K$ and with values in $\mathcal{F}(X, Y^{**})$, in such a way that

$$T(f) = \int f dm, \quad (f \in C(K, X)).$$

If $m : Bo(K) \rightarrow \mathcal{F}(X, Y)$ is a finitely additive measure, we shall denote by $|m|$ its semi-variation. One says that $|m|$ is continuous at $\emptyset$ if it has a control measure: a countably additive positive measure $\lambda$ on $Bo(K)$ such that

$$\lim_{\lambda(A) \to 0} |m|(A) = 0.$$  

1.10. Proposition. When $T \in W(C(K, X), Y)$, its associated representing measure $m$ is countably additive and verifies the following two conditions:

a) $|m|$ is continuous at $\emptyset$, and

b) for each $A \in Bo(K)$, $m(A) \in W(X, Y)$.

Thus, it seems natural to ask which properties pass from $T$ to $m$ and vice versa.
2. OPERATORS AND MEASURES

By mimicry of the proofs made in [3], [4] and [20] for the cases $p=1, \infty$ one can easily obtain:

2.1. Proposition. Let $T \in \mathcal{C}_p(C(K,X),Y)$, and let $m$ its representing measure. Then:

a) $|m|$ is continuous at $\emptyset$, and 

b) for each $A \in \mathcal{B}_0(K)$, $m(A) \in C_p(X,Y)$.

Nevertheless, these two conditions a) and b) do not characterize $C_p$ operators. In [1], there is shown an operator $T$ from $C([0,1],c_0)$ to $c_0$ which is not in $C_1$ but is such that its representing measure $m$ has continuous semi-variation at $\emptyset$, and $m(A)$ is a compact operator for any Borel set $A \subset [0,1]$.

2.2. Proposition. Let $T \in \mathcal{L}(C(K,X),Y)$ have a representing measure $m$ satisfying:

a) $|m|$ is continuous at $\emptyset$ and admits a discrete control measure, and 

b) for each $A \in \mathcal{B}_0(K)$, $m(A) \in C_p(X,Y)$.

Then $T \in \mathcal{C}_p(X,Y)$.

Since every Radon measure over a dispersed compact set is discrete (see [16, §2]), it follows that:

2.3. Corollary. If $K$ is dispersed and $T \in \mathcal{L}(C(K,X),Y)$ is such that its representing measure $m$ satisfies:

a) $|m|$ is continuous at $\emptyset$, and 

b) for each $A \in \mathcal{B}_0(K)$, $m(A) \in C_p(X,Y)$,

then $T \in \mathcal{C}_p(X,Y)$. 
Corollary (2.3) asserts that (2.1) is an equivalence when \( K \) is dispersed. We can also expect an equivalence when some condition is imposed on \( X \).

2.4. Proposition. Let \( 1 \leq p \leq \infty \). The following are equivalent:

a) \( \text{id}(X) \in C_p \).

b) Given any compact space \( K \) and any Banach space \( Y \), an operator \( T \in C_p(\mathcal{C}(K,X),Y) \) if and only if its representing measure satisfies

i) \( |m| \) is continuous at \( \emptyset \), and

ii) for each \( A \in \mathcal{B}_0(K) \), \( m(A) \in C_p \).

Concerning the dual scale of weakly-\( p \)-compact operators, we have:

2.5. Lemma. Let \( T \in \mathcal{L}(\mathcal{C}(K,X),Y) \) and \( p \geq 1 \). The following are equivalent (\( \hat{T} \) is the restriction to \( \mathcal{B}(\Sigma,X) \) of the operator \( T^{**} \)):

a) \( T \in W_p(\mathcal{C}(K,X),Y) \),  \( \hat{T} \in W_p(\mathcal{B}(\Sigma,X),Y) \),  c) \( T^{**} \in W_p(\mathcal{C}(K,X)^{**},Y) \).

Proof. Since \( T \in W(A,B) \) if and only if \( T^* \) (or any of its iterated duals) is weak*-to-weak continuous, and the unit ball of \( A \) is weak*-dense in the unit ball of \( A^{**} \), we have:

\[
T^{**}(B_A) = T^{**}(\overline{B_A^{\text{weak}^*}}) \subseteq \overline{T(B_A)}
\]

from which the result follows.

That immediately gives:

2.6. Proposition. Let \( T \in W_p(\mathcal{C}(K,X),Y) \), \( p \geq 1 \). Its associated measure verifies:
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a) $|m|$ is continuous at $\emptyset$, and
b) for each $A \in \mathcal{B}(K)$, $m(A) \in W_p(X,Y)$.

The converse is not true; see the comments after (2.1).

3. DUNFORD-PETTIS-LIKE PROPERTIES

A Banach space $X$ is said to have the Dunford-Pettis property if any weakly compact operator $T:X \rightarrow Y$ transforms weakly compact sets of $X$ into norm compact sets of $Y$. This property can be described by means of the inclusion $W(X,Y) \subseteq C_p(X,Y)$. We can weaken this requirement in the following manner:

3.1. Definition. Let $1 \leq p \leq \infty$. We shall say that a Banach space $X$ has the Dunford-Pettis property of order $p$ (in short $DPP_p$) if the inclusion $W(X,Y) \subseteq C_p(X,Y)$ holds for any Banach space $Y$.

Obviously $DPP_p$ implies $DPP_q$ when $q < p$. Also, $DPP = DPP_\infty$ and every Banach space has $DPP_1$. It follows from the definition that if $id(X) \in C_p$ then $X$ has $DPP_p$, and that if $id(X) \in W_p$ then $X$ does not have $DPP_p$. The following result contains analytical and geometrical characterizations of the $DPP_p$.

3.2. Proposition. For a given Banach space $X$, the following are equivalent:

a) $X$ has $DPP_p$ ($1 \leq p \leq \infty$).

b) If $(x_n)$ is a weakly-$p$-summable sequence of $X$ and $(x_n^*)$ is weakly null in $X^*$ then $(x_n^* x_n) \rightarrow 0$.

c) Every weakly compact operator $T:X \rightarrow Y$ transforms weakly-$p$-compact sets of $X$ into norm compact sets of $Y$. 
Proof. The proof of the equivalence between (a) and (b) is obtained as in [21]. We prove the equivalence of (a) and (c).

\((c) \Rightarrow (a)\): Consider \(T : X \to Y\) a weakly compact operator, and \((x_n)\) a weakly-\(p\)-summable sequence in \(X\). We form the set:

\[
\text{conv}_p((x_n)) = \{ \sum_{n=1}^\infty \lambda_n x_n : \lambda_n^p \leq 1 \}
\]

which we shall refer to as the \(p^*\)-convex hull of \((x_n)\). Clearly, \(\text{conv}_p((x_n))\), the continuous image by the natural operator associated to \((x_n)\) of the unit ball of \(l_{p^*}\), is a weakly-\(p\)-compact set. Since \(T \in C_p\) and \(l_{p^*} \in W_{p^*}\), \(T(\text{conv}_p((x_n)))\) is compact, and \((Tx_n)\) is norm-null.

\((a) \Rightarrow (c)\): If \(A\) is a weakly-\(p\)-compact set of \(X\), then for each bounded sequence \((z_m)\) of \(A\) there is a point \(z \in A\), and a sub-sequence \((z_{m_k})\), such that \((z_{m_k} - z)\) is weakly-\(p\)-summable. We set \((x_n) = (z_{m_k} - z)\), and apply to this sequence the preceding argument, to conclude that \((Tx_n)\) admits a norm null sub-sequence.

3.3. Examples. The following examples are immediate after (1.9). In fact, these results give the optimum values of \(p\).

a) \(C(K)\) and \(L_1\) have the \(DPP\), and therefore the \(DPP_p\) for all \(p\).

b) If \(1 < r < \infty\), \(l_r\) has the \(DPP_p\) for \(p < r^*\).

c) If \(1 < r < \infty\), \(L_r(y)\) has the \(DPP_p\) for \(p < \min(2, r^*)\).

d) Tsirelson's space \(T\) has \(DPP_p\) for all \(p < \infty\). However, since \(T\) is reflexive, it does not have \(DPP\).

e) Tsirelson's dual space \(T^*\) does not have \(DPP_p\) for any \(p > 1\).

Coming back to continuous vector function spaces, we have:
3.4. **Proposition.** If \( \text{id}(X) \in C_p \) then, for any compact \( K \), \( C(K,X) \) has \( DPP_p \).

**Proof.** Let \( T \in \mathcal{W}(C(K,X),Y) \). If \( (f_n) \) is a weakly-p-summable sequence in \( C(K,X) \), then for each \( t \in K \), the sequence \( (f_n(t)) \) is also weakly-p-summable in \( X \), and thus it is norm null. The sequence \( (Tf_n) \) is also null by [5, Th. 2.1].

3.5. **Corollary.** Given any compact space \( K \) and \( 1 < p < \infty \), \( C(K,l_p) \) has \( DPP_r \) for all \( r < p^* \); it does not have \( DPP_{p^*} \).

A "limit case" is provided by Tsirelson's spaces (compare this result with (3.13)):

3.6. **Corollary.** If \( T \) denotes Tsirelson's space then, given any compact space \( K \) and \( 1 < p < \infty \), \( C(K,T) \) has \( DPP_p \) but not \( DPP \).

Now, we see what happens if we replace the condition "\( \text{id}(X) \in C_p \)" by the weaker "\( X \text{ has the } DPP_p \)."

3.7. **Example.** Talagrand’s construction of a Banach space \( X \) having \( DPP \) but such that \( C(K,X) \) does not have \( DPP \) (see [22]), can be modified in such a form that we obtain Banach spaces \( T_p \) \( (p > 1) \) having \( DPP \), and such that \( C(K,T_p) \) does not have \( DPP_p \). Talagrand’s original example corresponds to \( T_2 \).

What can be said about \( C(K,X) \) when \( X \) simply has \( DPP_p \)? The following theory was developed in [4] and [2] for \( DPP \).

3.8. **Definition.** An operator \( T : C(K,X) \to Y \), whose associated measure \( m \) has continuous semi-variation at \( \emptyset \), is said to be almost-\( C_p \) if, for each weakly-p-summable sequence \( (x_n) \) of \( X \) and each bounded sequence \( (\phi_n) \) of \( C(K) \), the sequence \( T(\phi_n x_n) \) converges to 0 in \( Y \). Obviously, \( C_p \)-operators are almost-\( C_p \).
3.9. **Theorem.** The following are equivalent:

a) $X$ has DPP$_p$.

b) For each compact space $K$, every weakly compact operator $T:C(K,X) \to Y$ is almost-C$_p$.

c) Every weakly compact operator $T:C([0,1],X) \to Y$ is almost-C$_p$.

d) Every weakly compact operator $T:C([0,1],X) \to c_0$ is almost-C$_p$.

(The proof is exactly as [2, Th. 5]).

3.10. **Corollary** ([10, [13]). Let $1 \leq p \leq \infty$. For a dispersed compact space $K$, the following are equivalent:

a) $C(K,X)$ has DPP$_p$.

b) $X$ has DPP$_p$.

**Proof.** Implication a)$\Rightarrow$b) follows from (3.9). Conversely, if $T \in W(C(K,X),Y)$ with representing measure $m$, for each Borel set $A \subset K$, $m(A) \in W(X,Y) \subset C_p(X,Y)$, since $X$ has DPP$_p$. Applying (2.3), we obtain $T \in C_p$.

Concerning the scales $W_p$, Diestel and Seifert proved in [11] that weakly compact operators defined on $C(K)$ spaces are Banach-Saks operators. Recall that an operator $T \in \mathcal{S}(X,Y)$ is said to be Banach-Saks (in short $T \in BS$) if any bounded sequence $(x_n)$ of $X$ admits a sub-sequence $(x_{m_k})$ such that $(Tx_{m_k})$ has norm-convergent arithmetic means.

Núñez [18] extended this result to $C(K,X)$ spaces showing that, when $X$ is super-reflexive, then weakly compact operators defined on $C(K,X)$ are Banach-Saks. In [9], it is shown a vector measure whose range is not a weakly-$p$-compact set for any $p$. That example provides a weakly compact operator $T$, defined on a certain $C(K)$ space, which, for every $p$, does not belong to $W_p$, showing that, in general, $X \in W_p$ does not imply
W(C(K,X),Y)⊂W_p(C(K,X),Y), and therefore, that in some sense, the result of Diestel and Seifert cannot be improved.

Despite that negative result, when K is a dispersed compact space, some positive results can be obtained:

3.11. Proposition. If X∈W_p then W(c_0(X),Y)⊂W_p(c_0(X),Y).

Proof. Let T∈W(c_0(X),Y) and let (f_n) be a bounded sequence in c_0(X). Let ε>0. For each n∈N, a number p_n exists so that ∥f_n(k)∥≤ε2^n for k≥p_n.

We write f_n = f_n^d + f_n^l, where

f_n^l = (f_n(1),...,f_n(p_n-1),0,0,...)

and

f_n^d = (0,0,...,0,f_n(p_n),f_n+1,...).

Since ∥f_n^d∥→0, it is enough to see that T(f_n^l) admits a weakly-p-convergent sub-sequence. For each k∈N, there exists q_k such that w_p((f_n^l(k)-x_k)|_{n≤q_k})≤λ (the constant λ can be chosen uniformly [15]).

We determine inductively a sequence of indices (q_{δn}) as follows:

q_{δ0} = q_1 and q_{δ(n+1)} = max{q_k : k≤p(q_{δ(n)})}

so that p(q_{δ(n+1)})>p(q_{δ(n)}), and consider the sub-sequence f_n^l = f_n^{q_{δn}}.

We now write f_n^l = s_n + t_n where

s_n = (0,0,0,...,f_n(p_{q_1}),...f_n(p_{q_{δn}}),0,0,...),

t_n = (0,0,0,...,f_n(p_{q_1}),...f_n(p_{q_{δn}}),0,0,...),

so that it is the continuous image of a block basic sequence constructed against the canonical basis of c_0. We see that, passing to a sub-sequence if necessary, (T_t_n) converges to 0.
The sequence
\[
(z_n) = \begin{cases} 
  z_n(k) = f_n(k) & \text{if } k \leq p(q_{n(i+1)}), \\
  z_n(k) = 0 & \text{otherwise},
\end{cases}
\]
however, is the continuous image of (a part of) the summing basis \((e_1 + \ldots + e_n)_n\) of \(c_0\).

If we set \(x = (x_1, x_2, x_3, \ldots) \in l_p(X)\), we see, passing again to a subsequence if necessary, that \(\|Tz_n - T**x\| \leq 2^n\).

Finally, if \((\xi_n)\) is a finite sequence in the unit ball of \(l_p\), then
\[
\|\sum \xi_n (Tz_n - T**x)\| \leq \|\sum \xi_n (Tz_n - Tz_n + Tz_n - T**x)\|
\leq \|T\| \cdot \|\sum \xi_n (z_n - z_n)\| + 1 \leq \lambda \cdot |T| + 1,
\]
thus finishing the proof.

**Remark.** If the choice of indices indicated in the proof is not possible because the sequence \((p_n)\) does not go to infinity, then we would be working in a finite product space \(X^\omega\); if it is because the sequence of \(q_n\) stops at \(q\), then we shall follow the same reasoning as in the last part with the sub-sequence, \(f_{q^0}, f_{q^0+1}, \ldots\)

### 3.12. Theorem

Let \(K\) be a dispersed compact space and \(X \in W_p\). Then:
\[
W(C(K, X), Y) \subseteq W_p(C(K, X), Y).
\]
**Proof.** Let \(T \in W(C(K, X), Y)\) and let \((f_n)\) be a bounded sequence in \(C(K, X)\). By a standard argument we can assume \(K\) to be countable, \(K = \{t_1, t_2, \ldots\}\). Since \(m\) (the associated measure of \(T\)) has continuous semi-
variation at $\emptyset$, a $p_n$ exists for each $n \in \mathbb{N}$ such that, if we set $B_k=\{t_j; j \geq k\}$, then $|m| (B_k) \leq 2^n$.

Once more we write $f_n = f_n^d + f_n^i$ where $f_n^d$ converges to 0 and $f_n^i$ is eventually zero. Since $f_n^i$ is a bounded sequence in a space isomorphic to some $c_0(\mathbb{N},X)$, the proof of (3.11) applies.

3.13. Corollary. If $K$ is a dispersed compact space and $T^*$ denotes Tsirelson's dual space, then $W(C(K),T^*), Y) \subset W_p(C(K),T^*), Y)$ for all $p>1$.

A sufficient condition on $X$ which guarantees the inclusion $W(C(K),X) \subset W_p(C(K),X)$ is given by:

3.14. Theorem. If $X$ does not contain $c_0$ finitely represented, then

$$W(C(K),X) \subset W_2(C(K),X).$$

Proof. If $X$ does not contain $c_0$ finitely represented, then there is a $p>1$ such that $\mathcal{S}(C(K),X) = W(C(K),X) \subset \Pi_p(C(K),X)$ by [19]. But each $p$-summing operator sub-factorizes through an $L_p$-space, which gives $\Pi_p \subset W_2$ when $p \geq 2$, and thus for all $p$.

The hypothesis is not necessary: just consider Tsirelson's space $T^*$.

4. FINAL REMARKS AND FURTHER QUESTIONS

Results (3.12) and (3.14) suggest the following problems:

Problem K. Characterize the compacts $K$ such that for any Banach space $X$

$$W(C(K),X) \subset W_2(C(K),X).$$

Problem X. Characterize those Banach spaces $X$ such that for any compact $K$
Notice that the hypothesis of (3.14) is not necessary: if $K$ is dispersed, then $W(C(K), X) \subset W_p(C(K), X)$ for all $p > 1$ and $T$ is not, for any $p < \infty$, of cotype $p$.

An application could be the following conjecture, essentially due to Drewnowski: Is it true that $\mathcal{L}(l_p, X) = K(l_p, X)$? One implication is clear. To see the other, notice that $X \in C_2$ and $\mathcal{L}(l_p, X) = K(l_p, X)$ are equivalent. Since $C_2 \circ W_2 = K$, and since $X \in C_2$ implies $\mathcal{L}(l_p, X) = W(l_p, X)$, the question is whether a) Banach spaces $X \in C_2$ satisfy affirmatively Problem X, or b) the Stone-Cech compactification of $N \cap \mathbb{N}$, satisfies affirmatively Problem K.

Another unsolved question about the relationships between $T$ and $m$ is the following: Is it true that if $K$ is a dispersed compact, and, for every Borel set $A$, the operator $m(A) \in W$, then $T \in W$?

The example in [9] mentioned before (3.11) shows that the hypothesis "$K$ dispersed" cannot be removed.

Besides this, Núñez proved in [18] that if $T: C(K, X) \to Y$, $K$ is dispersed and, for every Borel set $A$, the operator $m(A) \in BS$, then $T \in BS$. The connection with Núñez's result is the following:

Obviously property $W_p$ implies the Banach-Saks property. Moreover, for $p > 1$, the $p$-Banach-Saks property is defined as follows: A Banach space $X$ is said to have the $p$-Banach-Saks property when each bounded sequence $(x_n)$ admits a sub-sequence $(x_{n_k})$ and a point $x$ such that $(x_{n_k} - x)$ is a $p$-Banach-Saks sequence, i.e., satisfies an estimate of the form

$$\left| \sum_{k=1}^n x_k \right| \leq C n^{1/p}$$

for some constant $C > 0$ and all $n \in \mathbb{N}$. It is also clear that property $W_p$ implies the $p^*$-Banach-Saks property. In [6] can be seen a proof that, conversely, the $p^*$-Banach-Saks property implies, for all $r > p$, the property $W_r$. Therefore, what this question is looking for is the extension of Núñez's result to the scale of $p$-Banach-Saks properties.
References

Dunford-Pettis-like Properties of Continuous...