On a Nonlinear Stationary Problem in Unbounded Domains

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ABSTRACT. We study existence and some properties of solutions of the nonlinear elliptic equation

\[ N(x, a(u)) Lu = f \]

in unbounded domains. The above model is not a variational problem. Our techniques involve fixed point arguments and Galerkin method.

1. INTRODUCTION

In this paper we present some results on the following problem:

\[
\begin{cases}
N(x, a(u)) Lu = f & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega = \Gamma
\end{cases}
\]

where \( \Omega \) is an open subset of \( \mathbb{R}^N \), \( N \geq 1 \) with boundary \( \Gamma \), \( N \) is a real valued function with domain \( \Omega \times [0, +\infty) \) and \( L \) is a differential operator defined by

\[
Lu = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i} \left( a_{ij} \frac{\partial u}{\partial x_j} \right) + a_u u
\]

and

\[
a_u = \sum_{i,j=1}^{N} \int_{\Omega} a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} dx + \int_{\Omega} a_u(x) u^2(x) dx
\]

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The origin of this problem can be found in Lions [6] where the following evolution problem connected with nonlinear vibrations was studied

\[
\begin{align*}
\frac{\partial u}{\partial t} - M(||u||^2) \Delta u &= f \quad \text{in } Q = \Omega \times (0, T) \\
u &= 0 \quad \text{on } \Sigma = \partial \Omega \times (0, T) \\
u(0) &= u_0, \quad u'(0) = u_1
\end{align*}
\]

(1.2)

In (1.2) \(M\) is a continuous real valued function which satisfies \(M(\lambda) \geq m_0 > 0, \forall \lambda \in \mathbb{R}\) and \(\Omega\) is an open bounded set of \(\mathbb{R}^N\) with smooth boundary \(\Gamma = \partial \Omega\).

Menzala [7] solved the problem (1.2) when \(\Omega = \mathbb{R}^N\) and the initial data are smooth functions.

Ebihara-Miranda-Medeiros [3] solved the problem (1.2) when \(\Omega\) is an open bounded set of \(\mathbb{R}^N\) with smooth boundary, \(M\) is a nonnegative continuously differentiable real function and the initial data are smooth.


In Lions [6] the following problem was also proposed:

\[
\begin{align*}
\frac{\partial u}{\partial t} - M(x, ||u||^2) \Delta u &= f \quad \text{in } Q \\
u &= 0 \quad \text{on } \Sigma \\
u(0) &= u_0, \quad u'(0) = u_1
\end{align*}
\]

(1.3)

This problem was solved in Rivera [8] when \(\Omega\) is a smooth open bounded set of \(\mathbb{R}^N\) and

\[M(x, \lambda) \geq m_0 > 0, \quad \forall x \in \Omega, \quad \forall \lambda \in \mathbb{R}\]

He used Galerkin method and the discrete spectrum of the Laplacian operator in bounded domains.

When \(\Omega\) is a general unbounded open set of \(\mathbb{R}^N\), in Vasconcellos [9] the existence and uniqueness of the solution for the problem (1.3) was proved. There, fixed point arguments together with energy methods were used.

The elliptic problem (1.1) can not be defined like a variational problem, that is, one can not find a nonlinear functional \(J\) defined in a suitable Sobolev space, such that the equation of problem (1.1) is its Euler equation. Then, to solve this problem we shall use a consequence of Brouwer fix point theorem and Galerkin methods.
In the section 2 we prove an abstract existence theorem (Theorem 2.1). Some properties about the set of solutions are given.

In section 3 we apply the Theorem 2.1 to solve the problem (1.1) when \( \Omega \) is a general open subset of \( \mathbb{R}^N \).

Our assumptions about the mapping \( N \) and the operator \( L \) are described in §3.

2. AN ABSTRACT EXISTENCE THEOREM

Let \( H \) be a real separable Hilbert space with inner product denoted by \( \langle \cdot, \cdot \rangle \) and norm \( |\cdot| \).

We consider a linear operator \( A \) in \( H \), with the following properties:

(2.1) \( D(A) \), the domain of \( A \), is dense in \( H \).

(2.2) \( A \) is a self-adjoint operator and there is a constant \( C_0 > 0 \) such that
\[
\langle Au, u \rangle \geq C_0 |u|^2,
\]
for each \( u \) in \( H \).

(2.3) The Hilbert space \( V = D(A^{1/2}) \), with norm denoted by \( |v| = |A^{1/2}v| \), for \( v \) in \( V \), is compactly embedded on \( H \).

Now, we denote by \( \mathcal{L}^b(H) \) the space of the symmetric linear bounded operators in \( H \) with the whole space as domain and we define the mapping
\[
M(\lambda) = M(\lambda)u, \quad \text{for each } u \text{ in } H\text{ and } \lambda \geq 0.
\]

Theorem 2.1 (abstract existence theorem).

Under above hypotheses, if \( f \) belongs to \( H \), then there exists \( u \) in \( D(A) \) such that:

(2.6) \( M(|u|^2)Au = f \), in \( H \).

To prove this theorem, it is sufficient to show that, for \( f \) in \( H \) there exists \( v \) in \( V \) such that:

(2.7) \( (M(|v|^2)A^{1/2}v, A^{1/2}w) = (f, A^{1/2}w) \), \quad \forall w \in V \)
In fact, if $v \in V$ satisfies (2.7), we define $u = A^{-1/2} v$. Then, $u$ belongs to $D(A)$ and $\|u\|^2 = |v|^2$. Moreover, from (2.7) we obtain that:

$$(M(\|u\|^2)Au|A^{1/2} \omega) = (f|A^{1/2} \omega), \quad \forall \omega \in V$$

Since, $\omega = A^{-1/2} z$ belongs to $V$ for every $z$ in $H$ we obtain (2.6).

Now, let us prove (2.7).

By (2.5) we obtain that

$$(2.8) \quad (M(\|v\|^2)A^{1/2} v|A^{1/2} v) - (f|A^{1/2} v) \geq m_v \|v\|^2 - \|f\| \|v\| \quad \forall v \in V$$

By (2.3), we observe that the operator $A$ has discrete spectrum, i.e., there exists a sequence of real numbers $\{\lambda_\nu\}$ and a sequence $\{e_n\}$ of elements in $D(A)$ such that:

$$(2.9) \quad 0 < \lambda_\nu \leq \lambda_{\nu+1}, \quad \nu = 1, 2, ...$$

$$(2.10) \quad \lim_{\nu \to +\infty} \lambda_\nu = +\infty$$

$$(2.11) \quad \Lambda \omega_\nu = \lambda_\nu \omega_\nu$$

$$(2.12) \quad \{\omega_1, \omega_2, ...\} \text{ is an orthonormal complete set in } H.$$ We denote by $V_\nu$ the space generated by $\{\omega_1, \omega_2, ..., \omega_\nu\}$, $\nu = 1, 2, ...$ and we define $P: \mathbb{R}^r \to \mathbb{R}^r$ by $P(\alpha_1, ..., \alpha_\nu) = (\beta_1, ..., \beta_\nu)$, where

$$\beta_j = \left(M(\|v_\nu\|^2)A^{1/2} v_\nu|A^{1/2} \frac{\alpha_j}{\sqrt{\lambda_j}} \right) - \left(f|A^{1/2} \frac{\omega_j}{\sqrt{\lambda_j}} \right), \quad j = 1, ..., \nu$$

and

$$v_\nu = \sum_{j=1}^\nu \alpha_j \lambda_j^{-1/2} \omega_j.$$ By (2.4) and (2.8) we obtain that: $P$ is a continuous function which satisfies

$$(2.13) \quad (P\alpha, \alpha) \geq 0 \quad \forall \alpha \in \mathbb{R}^r \text{ such that } |\alpha| = \left|\frac{f}{m_v}\right|$$

where $|\alpha| = \left(\sum_{j=1}^\nu \alpha_j^2\right)^{1/2}$ and $(\beta, \alpha)$ is a dot product in $\mathbb{R}^r$.

Now, we recall the following lemma:
Lemma 2.1 Let $P: \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a continuous function which satisfies:
$q \geq 0$ such that $(Pa, a) \geq 0, \forall a \in \mathbb{R}^n, |a| = r$. Then there exists $a_0 \in \mathbb{R}^n$
$|a_0| \leq r$ such that $P(a_0) = 0$.

This lemma is a consequence of the Brouwer point fixed theorem and one

So, by (2.13) and Lemma 2.1 we obtain for each $\nu \in \mathbb{N}$ a vector $v_\nu$ in $V$, such that:

\begin{align*}
(2.14) & \quad \|v_\nu\| \leq \frac{1}{m_\nu} |f|, \quad \nu = 1, 2, \ldots . \\
(2.15) & \quad (M(|v_\nu|^2)A^{1/2}v_\nu|A^{1/2}a_j \frac{a_j}{\sqrt{\lambda_j}}) = (fA^{1/2}a_j \frac{a_j}{\sqrt{\lambda_j}}) \quad j = 1, \ldots , \nu
\end{align*}

By (2.14) and assumption (2.3) there is a subsequence, also denoted by

\begin{align*}
(2.16) & \quad \lim_{\nu \to \infty} v_\nu = v \text{ weak in } V. \\
(2.17) & \quad \lim_{\nu \to \infty} v_\nu = v \text{ strong in } H.
\end{align*}

By (2.16), (2.17) and (2.4) we fix $j$ and passing the limit in (2.15) we obtain

Remark 2.1 If $u$ satisfies (2.6) it follows, by (2.5) that

\begin{align*}
(2.18) & \quad |Au| \leq \frac{|f|}{m_\nu} \\
\end{align*}

and therefore by (2.2)

\begin{align*}
(2.19) & \quad \|u\| \leq \frac{|f|}{\sqrt{c_0}m_\nu}
\end{align*}

Theorem 2.2 (Properties of the solutions).

Under the hypotheses of the Theorem 2.1, if $f$ belongs to $H$ then:

\begin{align*}
(2.20) & \quad \text{The set } \chi_f = \{u \in D(A): M(\|u\|^2)Au = f \} \text{ is a compact set of } V.
\end{align*}
(2.21) \( \lim_{f \to f_0} d_V(x_f, x_{f_0}) = 0 \) where \( d_V(x_f, x_{f_0}) = \inf \{ \| u - u_0 \| : u \in x_f, u_0 \in x_{f_0} \} \).

(2.22) If \( M(\lambda) = \psi(\lambda) I_H \), where \( I_H \) is the identity operator on \( H \) and \( \psi: R \to [m_\psi, +\infty) \) is a continuous increasing function, then for each \( f \in H \) the set \( x_f \) is a singleton.

To prove (2.20) we consider \( \{ u_\nu \} \) a sequence in \( x_f \), then by (2.19), there is a subsequence of \( \{ u_\nu \} \), also denoted by \( \{ u_\nu \} \) such that:

\[
\lim_{\nu \to \infty} A u_\nu = A u \text{ weak in } H, \text{ where } u \text{ belongs to } D(A).
\]

Hence, by (2.3),

\[
\lim_{\nu \to \infty} u_\nu = u \text{ strong in } V.
\]

Since that \( (M(\| u_\nu \|^2) A u_\nu) = (f_\nu f) \) for all \( \nu \in H \) and for each \( \nu \in N \), then by assumptions about \( M \) we obtain that \( u \) belongs to \( x_f \).

To prove (2.21), let \( f_\nu \) be in \( H \) and we consider \( \{ f_\nu \} \) a sequence in \( H \) such that:

\[
\lim_{\nu \to \infty} f_\nu = f_\infty \text{ strong in } H.
\]

For each \( \epsilon > 0 \) we define \( P_\epsilon = \{ \nu \in N : d_V(x_{f_\nu}, x_{f_\infty}) \geq \epsilon \} \). We claim that \( P_\epsilon \) is a finite set.

In fact, suppose that \( P_\epsilon \) is an infinite set, then there is \( \sigma: N \to N \) strictly increasing function such that \( \sigma(N) = P_\epsilon \).

We denote by \( g_\nu = f_{\sigma(\nu)}, \forall \nu \in N \) thus we obtain, for each \( \nu \in N, u_\nu \in x_{f_\nu} \) and therefore by (2.19) \( \{ A u_\nu \} \) is a bounded sequence in \( H \).

Hence there is a subsequence of \( \{ u_\nu \} \) (also denoted by \( \{ u_\nu \} \)) and \( v \) in \( D(A) \) such that:

(2.23) \( \lim_{\nu \to \infty} A u_\nu = A v \text{ weak in } H \).

(2.24) \( \lim_{\nu \to \infty} u_\nu = v \text{ strong in } V \).

Since \( (A u_\nu | M(\| u_\nu \|^2) = (g_\nu, \omega), \forall \omega \in V \text{ and } \nu \in N \) passing to limit we obtain that:

\[
(A v | M(\| v \|^2)) = (f_\infty, \omega), \quad \forall \omega \in V.
\]
Therefore, \( v \) belongs to \( \chi_{f_{\cdot}} \) and moreover \( d_{\nu}(x_{f_{\cdot}}, x_{f_{\cdot}}) \leq \|u_{\nu} - v\|, \nu \in \mathbb{N} \). This proves (2.21) by contradiction.

To prove (2.22) we consider \( \varphi \) belonging to \( V \) and we define the following functional:

\[
J(\varphi) = \frac{1}{2} \int_{0}^{[b(x)]} \psi(\lambda) d\lambda - (f | \varphi).
\]

where \( f \) belongs to \( H \).

Let \( u \) be in \( \chi_{f} \) then by (2.25) we obtain that:

\[
\lim_{t \to 0} \frac{J(u + t(\varphi - u)) - J(u)}{t} = 0, \text{ for all } \varphi \text{ in } V.
\]

Now, since that \( g(t) = J(u + t(\varphi - u)) \) is a differentiable convex function on \( \mathbb{R} \) we obtain that:

\[
\frac{g(t) - g(0)}{t} < g(1) - g(0), \quad 0 < t < 1
\]

therefore by (2.26)

\[
0 = g'(0) \leq g(1) - g(0)
\]

so, \( J(u) \leq J(\varphi), \forall \varphi \in V \) that is, if \( u \in \chi_{f} \), we prove that

\[
J(u) = \min \{ J(\varphi); \varphi \in V \}.
\]

Since \( J \) is strictly convex functional we have (2.22).

3. APPLICATION TO A NONLINEAR PROBLEM

First, we need to present a compact immersion result for unbounded domains, in weighted Sobolev spaces.

Let \( \Omega \) be an unbounded set of \( \mathbb{R}^{N} \) with the cone property and \( q: \Omega \to \mathbb{R} \) is positive measurable function which satisfies the following properties

\[
q \in L_{\text{loc}}^{\infty} (\Omega)
\]

\[
\lim_{|x| \to \infty} q(x) = 0
\]
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Let $L^2(\Omega, q)$ be the Hilbert space of measurable functions $u: \Omega \to \mathbb{R}$ such that $\int_{\Omega} |u(x)|^2 q(x) \, dx < +\infty$ endowed with the norm

$$|u| = \left( \int_{\Omega} |u(x)|^2 q(x) \, dx \right)^{1/2}$$

**Remark 3.1** Since the linear application $\sigma: L^2(\Omega, q) \to L^2(\Omega)$ defined by $\sigma(u) = u \sqrt{q}$ is an isometric isomorphism, then $L^2(\Omega, q)$ is a separable Hilbert space with inner product $(u|v) = \int_{\Omega} u(x)v(x)q(x) \, dx$ for $u, v \in L^2(\Omega, q)$.

**Proposition 3.1** (compact immersion).

The Sobolev space $H^1(\Omega)$ is compactly embedded in $L^2(\Omega, q)$.

This proposition is proved in Benci-Fortunato [1] corollary 2.10.

**Remark 3.2** It is possible to consider, in the above proposition, $\Omega$ a general open unbounded set of $\mathbb{R}^n$, if we replace the assumption (3.2) by

$$\lim_{x \to x_0} q(x) = 0 \quad x_0 \in \partial \Omega \text{ and } \lim_{|x| \to +\infty} q(x) = 0.$$  

In this case we can see the corollary 2.9 in Benci-Fortunato [1].

Let $N: \Omega \times [0, +\infty) \to \mathbb{R}$ be a function satisfying the following assumptions:

(3.5) $N(x, \lambda): [0, +\infty) \to \mathbb{R}$ is a continuous function a.e. in $x \in \Omega$.

(3.6) $N(\cdot, \lambda): \Omega \to \mathbb{R}$ belongs to $L^\infty(\Omega)$, $\forall \lambda \geq 0$.

(3.7) $N \in L^\infty_{\text{loc}}([0, +\infty) \times (\Omega))$.

(3.8) There exists $m_0 > 0$ such that $N(x, \lambda) \geq m_0$, $\forall (x, \lambda) \in \Omega \times [0, +\infty)$.

We consider $H = L^2(\Omega, q)$ where $q$ satisfies (3.1) and (3.2) (or (3.4)).

Now, we define, $\forall \lambda \geq 0$, the operator $M(\lambda)$ in $H$ by

$$M(\lambda)u = N(\cdot, \lambda)u, \quad u \in H.$$  

Then, $\forall \lambda \geq 0$, $M(\lambda) \in \mathcal{L}(H)$ and by (3.5)–(3.8), the mapping $M: [0, +\infty) \to \mathcal{L}(H)$ satisfies (2.4) and (2.5).
We consider \( a_{ij}, i, j = 1, \ldots, n \) and \( a_\alpha \) elements of \( L^\infty(\Omega) \) which satisfy:

\[ a_{ij}(x) \geq c_\alpha(x) \quad \text{for all } x \in \Omega, \]

where \( c_\alpha \) is a nonnegative constant.

(3.10) There is \( c_\alpha > 0 \) such that \( \sum_{i,j=1}^n a_{ij}(x) \zeta_i \zeta_j \geq c_\alpha (\zeta_1^2 + \cdots + \zeta_n^2) \) a.e. in \( \Omega \) and \( a_\alpha(x) \geq c_\alpha \) a.e. in \( \Omega \).

(3.11) \( a_{ij} = a_{ji}, \quad i, j = 1, \ldots, n. \)

We define the differential operator \( Lu = -\sum_{i,j=1}^n \frac{\partial}{\partial x_j} \left( a_{ij} \frac{\partial u}{\partial x_i} \right) + a_\alpha u \), a.e. in \( \Omega \) and the bilinear form \( a: H^1_0(\Omega) \times H^1_0(\Omega) \rightarrow \mathbb{R} \) by

\[
a(u,v) = \sum_{i,j=1}^n \int_{\Omega} a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} \, dx + \int_{\Omega} a_\alpha(x) u(x) v(x) \, dx.
\]

By (3.10) and (3.11) \( a(\cdot, \cdot) \) is a symmetric, coercive, continuous bilinear form.

We define \( D(A) = \{ u \in H^1_0(\Omega) : \frac{1}{q} Lu \in L^2(\Omega, \mu) \} \). Then, by (3.10), (3.11) and Proposition 3.1, the operator \( A: D(A) \subset H \rightarrow H \) defined by \( Au = \frac{1}{q} Lu \), \( u \in D(A) \), satisfies (2.1), (2.2) and (2.3), where \( V = D(A^{1/2}) = H^1_0(\Omega) \) with the norm \( \| v \|_V = \| A^{1/2} v \| = [a(v, v)]^{1/2} = [a(v)]^{1/2}. \)

Now, by Theorem 2.1, if \( f \) belongs to \( H = L^2(\Omega, \mu) \), there exists \( u \) in \( H^1_0(\Omega) \) such that:

\[ N(x, a(u)) Lu = q f(x) \quad \text{a.e. in } \Omega. \]

(3.12)

Therefore, if \( f \) belongs to \( L^2(\Omega, \frac{1}{q}) \), by (3.12) we obtain that, there exists \( u \in H^1_0(\Omega) \) such that:

\[ N(x, a(u)) Lu = f(x) \quad \text{a.e. in } \Omega. \]

(3.13)

Remark 3.2 If we consider \( q \in L^\infty(\Omega) \) we obtain that \( L^2(\Omega, \frac{1}{q}) \) is contained in \( L^2(\Omega) \). In general, if \( Q \) is the set of \( q \in \Omega \rightarrow \mathbb{R} \) which belong to \( L^\infty(\Omega) \) and satisfy (3.2) (or (3.4)) then \( X = \bigcup_{q \in Q} L^2(\Omega, \frac{1}{q}) \) is a dense subset of \( L^2(\Omega) \).

Remark 3.3 If \( \Omega \) is an open bounded subset of \( \mathbb{R}^N \), we consider \( H = L^2(\Omega) \) and the problem (1.1) has a solution \( u \) in \( H^1_0(\Omega) \) for each \( f \) in \( L^2(\Omega) \).
Remark 3.4 If we consider $N: [0, +\infty) \to [m_0, +\infty)$ increasing continuous function, then by Theorem 2.2, the problem

$$\begin{align*}
N(a(u)) \Delta u &= f & \text{in } \Omega \\
u &= 0 & \text{on } \partial \Omega
\end{align*}$$

has an unique solution $u$ in $H^1_0(\Omega)$, for each $f \in H$.

Where $H = L^2(\Omega, \frac{1}{q})$ if $\Omega$ is an open unbounded set or $H = L^2(\Omega)$ if $\Omega$ is an open bounded set.

Remark 3.5 All the solutions of the problem (1.1) have the properties given by the Theorem 2.2.

References