On Slice Knots in the Complex
Projective Plane

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ABSTRACT. We investigate the knots in the boundary of the punctured complex projective plane. Our result gives an affirmative answer to a question raised by Suzuki. As an application, we answer to a question by Mathieu.

1. INTRODUCTION

Throughout this paper, we work in the smooth category, all manifolds are oriented and all the homology groups are with integral coefficients.

Let $M$ be a closed 4-manifold, $B^4$ an embedded 4-ball in $M$, and $K$ a knot in $\partial (M - \text{Int } B^4)$. If $K$ bounds a properly embedded 2-disk in $M - \text{Int } B^4$ then we call the knot $K$ a slice knot in $M$. Let $\text{Slice}(M)$ be the set of slice knots in $M$. It is well-known that $\text{Slice}(S^4)$ is proper subset of the set of knots (Fox and Milnor [3]) and $\text{Slice}(S^4)$ is a subset of $\text{Slice}(M)$. In [17], Suzuki proved that $\text{Slice}(S^2 \times S^2)$ is equal to the set of knots, and asked the following question.

Question 1. Is there a 4-manifold $M$ such that $\text{Slice}(S^4)$ is a proper subset of $\text{Slice}(M)$ and $\text{Slice}(M)$ is a proper subset of the set of knots?

In [20], the author has proved that $\text{Slice}(CP^2)$ does not contain a $(-2,15)$-torus knot. This assertion gives an affirmative answer to Question 1 since $\text{Slice}(S^4)$ is a proper subset of $\text{Slice}(CP^2)$ (Kervaire and Milnor [6]). In [20], the author could not find a knot that belongs to neither $\text{Slice}(CP^2)$ nor $\text{Slice}(CP^3)$. In Section 2, we show that there exist the knots that belongs to neither $\text{Slice}(CP^2)$ nor $\text{Slice}(CP^3)$.

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Let $K$ be a knot in $\partial(n_1 CP^2 \# n_2 \overline{CP}^2 - \text{Int} \, B^4)$. The knot $K$ is an evenly slice knot in $n_1 CP^2 \# n_2 \overline{CP}^2$ if $K$ bounds a properly embedded 2-disk in $n_1 CP^2 \# n_2 \overline{CP}^2 - \text{Int} \, B^4$ that represents an element $z(\varepsilon_1 \gamma_1 + \cdots + \varepsilon_n \gamma_n + \tilde{\varepsilon}_1 \tilde{\gamma}_1 + \cdots + \tilde{\varepsilon}_n \tilde{\gamma}_n)$ in $H_2(n_1 CP^2 \# n_2 \overline{CP}^2 - \text{Int} \, B^4, \partial)$, where $\gamma_1, \ldots, \gamma_n, \tilde{\gamma}_1, \ldots, \tilde{\gamma}_n$ are standard generators of $H_2(n_1 CP^2 \# n_2 \overline{CP}^2 - \text{Int} \, B^4, \partial)$, $\varepsilon_i = \pm 1, \tilde{\varepsilon}_i = \pm 1$ and $z$ is an integer. Let $e$-Slice $(n_1 CP^2 \# n_2 \overline{CP}^2)$ be the set of evenly slice knots in $n_1 CP^2 \# n_2 \overline{CP}^2$. (Note that $e$-Slice $(CP^2) = \text{Slice} (CP^2)$ and $e$-Slice $(\overline{CP}^2) = \text{Slice} (\overline{CP}^2)$.) In Section 3, we deal with the case $n_1 = n_2 = 1$ or $n_1 = 0$.

Let $K_0$ be a knot and $D^2$ a 2-disk intersecting transversely $K_0$ with the linking number $lk(\partial D^2, K_0) = l$. Let $p$ be a positive integer and $\varepsilon = \pm 1$. By performing $\frac{p}{\varepsilon}$-Dehn surgery along $\partial D^2$, we have a new knot. The new knot is said to be the knot obtained from $K_0$ by an $(ep, \tilde{l})$-twisting. Let $\mathcal{K}_p$ be the set of knots obtained from a trivial knot by an $(ep, \tilde{l})$-twisting for some integer $l$ and $\varepsilon = \pm 1$. Section 4 is devoted to two applications. Our first application is to find infinitely many knots that give a negative answer to the following question given by Mathieu [12].

**Question 2.** For any knot $K$, is there a positive integer $p$ such that $K \in \mathcal{K}_p$?

Our second one is to find infinitely many counterexamples to the following conjecture made by Akbulut and Kirby.

**Conjecture.** If $K$ is a knot with Arf invariant zero, then $K$ is obtained from a slice knot by a $(\pm 1, \pm 1)$-twisting. (Problem 1.46 (B) of [9].)

It is shown that a $(2, 7)$-torus knot cannot be obtained from a ribbon knot by a $(1,1)$-twisting by using Donaldson's outstanding theorem [1, Theorem 1] (see [10]). Since then Donaldson improved this result to drop "simply connectedness assumption" [2, Theorem 1], a $(2, 7)$-torus knot cannot be obtained from a slice knot by a $(1,1)$-twisting. Here we give infinitely many counterexamples in different knot cobordism classes.

Similar results for Question 2 were obtained independently by Katura Miyazaki [13].

1. **PRELIMINARIES**

In this section we introduce some useful lemmas to us. In particular, Lemmas 1.8 and 1.11 are key lemmas in this paper.
Let $\alpha, \beta$ be the standard generators of $H_2(S^3 \times S^3)$ with $\alpha^2 = \beta^2 = 0$, $\alpha \cdot \beta = 1$ and let $\gamma$ or $\gamma_i$ (resp. $\tilde{\gamma}$ or $\tilde{\gamma}_i$) be the standard generator of $H_2(\mathbb{C}P^2)$ (resp. $H_2(\mathbb{C}P^2)$) with $\gamma^2 = \gamma_i^2 = 1$ (resp. $\tilde{\gamma}^2 = \tilde{\gamma}_i^2 = -1$). From now on a homology class in $H_2(M - \text{Int} B^4, \partial)$ is identified with its image by the homomorphism

$$H_2(M - \text{Int} B^4, \partial) \cong H_2(M - \text{Int} B^4) \to H_2(M).$$

Let $l$ and $m$ be nonnegative integers and $e = \pm 1$. An $(el, m)$-torus link is the link that wraps around the standardly embedded solid torus in $S^3$ in the longitudinal direction $l$ times and in the meridional direction $m$ times, where the intersection number of the meridian and longitude is $e$. When $l$ and $m$ are relatively prime, it is a knot and called an $(el, m)$-torus knot. An $(el, m)$-torus knot is denoted by $T(el, m)$.

Let $L$ be a $\mu$-component link in $S^3$. Let $f_i : I \times I - S^3$, $i = 1, \ldots, m-1 (m \leq \mu)$ be mutually disjoint embeddings such that

(i) $f_i(I \times I) \cap L = f_i(I \times \partial I)$ for each $i = 1, \ldots, m-1$ and

(ii) the link $L' = C(L \cup \bigcup f_i(\partial I \times I) - \bigcup f_i(I \times \partial I))$ has the orientation compatible with that of $L - \bigcup f_i(I \times \partial I)$ and $\bigcup f_i(\partial I \times I)$.

The link $L'$ is said to be the link obtained from $L$ by $m$-fusion if the number of the components of $L'$ is $\mu - m$. In particular if the number of the components of $L'$ is one, then $L'$ is said to be the knot obtained from $L$ by complete fusion. We call the images $f_1(I \times I), \ldots, f_m(I \times I)$ the strips connecting the link $8(UA_1)$ such that $A = A_1 \cup \cdots \cup A_{m-1} + 2U \cup b_1 \cup \cdots \cup b_{2x+1}$ is an embedded 2-disk in $S^3 \times S^3 - \text{Int} B^4$ and $\partial A \subset \partial (S^3 \times S^3 - \text{Int} B^4)$ is a Figure 1. Since a $(2x, 4x)$-torus link is obtained from $\partial (\cup \Delta_i)$ by 2x-fusion, there exist $2x+1$ strips $b_1, \ldots, b_{2x+1}$ connecting the link $\partial (\cup \Delta_i)$ such that $\Delta = \Delta_1 \cup \cdots \cup \Delta_{2x+1} \cup b_1 \cup \cdots \cup b_{2x+1}$ is an embedded 2-disk in $S^2 \times S^2 - \text{Int} B^4$ and $\partial \Delta \subset (S^3 \times S^2 - \text{Int} B^4)$ is $-K'$.

1.1. **Lemma.** For any knot $K \in \mathcal{T}_{x}$, there exists an embedded 2-disk $\Delta$ in $S^2 \times S^2 - \text{Int} B^4$ such that $\Delta$ represents an element $2\alpha + 2x\beta$ in $H_2(S^2 \times S^2 - \text{Int} B^4, \partial)$ and $\partial \Delta \subset \partial (S^2 \times S^2 - \text{Int} B^4)$ is $-K'$.

**Proof.** We first deal with the case that $K \in \mathcal{T}_{x}$. It is easily seen that there exist mutually disjoint $2x + 2$ properly embedded 2-disks $\Delta_1, \ldots, \Delta_{2x+2}$ in $S^2 \times S^2 - \text{Int} B^4$ such that $\bigcup \Delta_i$ represents an element $2\alpha + 2x\beta$ and $\partial (\bigcup \Delta_i) \subset \partial (S^2 \times S^2 - \text{Int} B^4)$ is a Figure 1. Since a $(-2,4x)$-torus link is obtained from $\partial (\bigcup \Delta_i)$ by 2x-fusion, there exist $2x+1$ strips $b_1, \ldots, b_{2x+1}$ connecting the link $\partial (\bigcup \Delta_i)$ such that $\Delta = \Delta_1 \cup \cdots \cup \Delta_{2x+2} \cup b_1 \cup \cdots \cup b_{2x+1}$ is an embedded 2-disk in $S^2 \times S^2 - \text{Int} B^4$ and $\partial \Delta \subset (S^3 \times S^2 - \text{Int} B^4)$ is $-K'$.

The above argument remains valid in case $K \in \mathcal{T}_{x}$.
1.2. Lemma. For any knot $K \in \mathcal{T}_{x}$, there exists an embedded 2-disk $\Delta$ in $CP^2 \# CP^2 - \text{Int } B^4$ such that $\Delta$ represents an element $(2x + c) \gamma + (2x - c) \tilde{\gamma}$ in $H_2(CP^2 \# CP^2 - \text{Int } B^4, \partial)$ and $\partial \Delta \subset \partial(CP^2 \# CP^2 - \text{Int } B^4)$ is $-K$.

Proof. We first deal with the case that $K \in \mathcal{T}_x$. Let $O_j \cup O_{-j}$ be a 2-component trivial link in $\partial B^4$ such that $O_j$ is framed by $j$ (j = ±1). By considering the "Kirby's calculus"[8] as Figure 2, we note that there exist mutually disjoint $2x+1$ properly embedded 2-disks $\Delta_1, \ldots, \Delta_{2x+1}$ in $CP^2 \# CP^2 - \text{Int } B^4$ such that $\bigcup \Delta_i$ represents an element $(2x + 1) \gamma + (2x - 1) \tilde{\gamma}$ in $H_2(CP^2 \# CP^2 - \text{Int } B^4, \partial)$ and $\partial(\bigcup \Delta_i) \subset \partial(CP^2 \# CP^2 - \text{Int } B^4)$ is as Figure 3. Since a $(-2,4x)$-torus link is obtained from $\partial(\bigcup \Delta_i)$ by $(2x - 1)$-fusion, there exist $2x$ strips $b_1, \ldots, b_{2x}$ connecting the link $\partial(\bigcup \Delta_i)$ such that $\Delta = \Delta_1 \cup \ldots \cup \Delta_{2x+1} \cup b_1 \cup \ldots \cup b_{2x}$ is an embedded 2-disk in $CP^2 \# CP^2 - \text{Int } B^4$ and $\partial \Delta \subset \partial(CP^2 \# CP^2 - \text{Int } B^4)$ is $-K$.

By considering the Kirby's calculus as in Figure 4, the above argument remains valid in case $K \in \mathcal{T}_x$.

1.3. Lemma. (Rohlin [16]) Let $M$ be a connected, simply connected, closed 4-manifold. If $\xi \in H_2(M)$ is represented by an embedded 2-sphere in $M$, then
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Figure 2

(a) \( \left| \frac{\xi^2}{2} - \sigma(M) \right| \leq \text{rank } H_2(M) \) if \( \xi \) is divisible by 2,

(b) \( \left| \frac{\xi^2(q^2 - 1)}{2q^2} - \sigma(M) \right| \leq \text{rank } H_2(M) \) if \( \xi \) is divisible by an odd prime integer \( q \), where \( \sigma(M) \) is the signature of \( M \).

1.4. Lemma. (Weintraub [18], Yamamoto [19]) Let \( K \) be a knot. If the unknotting number of \( K \) is less than or equal to \( u \) then there exists embedded 2-disk \( \Delta \) in \( u(CP^2 \# CP^2) - \text{Int} \, B^4 \) such that \( \Delta \) represents the zero element in \( H_2(u(CP^2 \# CP^2) - \text{Int} \, B^4, \partial) \) and \( \partial \Delta \subset \partial (u(CP^2 \# CP^2) - \text{Int} \, B^4) \) is \( -K^1 \).

1.5. Lemma. (Lawson [11]) Let \( \xi \in H_2(CP^2 \# 2CP^2) \) be a characteristic element. The element \( \xi \) is represented by a 2-sphere in \( CP^2 \# 2CP^2 \) if and only if \( \xi^2 = -1 \).

1.6. Lemma. (Lawson [11]) Let \( \xi \in H_2(CP^2 \# nCP^2) \) (\( n \geq 3 \)) be a characteristic element. If \( \xi \) is represented by a 2-sphere in \( CP^2 \# nCP^2 \) then \( \xi^2 \leq -2 \).
Figure 3

Figure 4
1.7. Lemma. (Kikuchi [7]) Let \( \xi \in H_2(CP^2\# 3\overline{CP^2}) \) be a characteristic element. The element \( \xi \) is represented by a 2-sphere in \( CP^2\# 3\overline{CP^2} \) if and only if \( \xi^2 = -2 \).

1.8. Lemma. Let \( p \) be a positive integer and \( x \) a nonnegative integer. Let \( K \in \mathcal{T} \) be a knot such that the unknotting number of \( K \) is less than or equal to \( u \). If \( K \in e\text{-}
Slice(p \overline{CP^2}) \) then there exists an integer \( z \) such that \( z \) satisfies a condition

\[
\frac{8x-4}{p} \leq z^2 \leq \frac{4u}{p} + 4 \quad \text{and } z \text{ is even, or}
\]

\[
z^2 = 8x + 1 \quad \text{if } p = 1,
\]

\[
z^2 = 4x + 1 \quad \text{if } p = 2,
\]

\[
\frac{8x+2}{p} \leq z^2 \leq \frac{9}{2} \left( \frac{u}{p} + 1 \right) \quad \text{and } z \text{ is odd if } p \geq 3.
\]

Proof. Suppose that \( K \in \mathcal{T} \cap e\text{-}
Slice(p \overline{CP^2}) \) and the unknotting number of \( K \) is less than or equal to \( u \). Since \( K \in \mathcal{T} \cap e\text{-}
Slice(p \overline{CP^2}) \), there exists an integer \( z \) such that

1. \( 2x + 2x + z(\tilde{e}_1 \tilde{\gamma}_1 + \ldots + \tilde{e}_p \tilde{\gamma}_p) \in H_2(S^2 \times S^2 \# p \overline{CP^2}) \) is represented by a 2-sphere in \( S^2 \times S^2 \# p \overline{CP^2} \) and

2. \( (2x+1) \gamma + (2x-1) \tilde{\gamma} + z(\tilde{e}_1 \tilde{\gamma}_1 + \ldots + \tilde{e}_p \tilde{\gamma}_p) \in H_2(CP^2 \# (p+1) \overline{CP^2}) \) is represented by a 2-sphere in \( CP^2 \# (p+1) \overline{CP^2} \),

by Lemmas 1.1, 1.2 and the definition of evenly slice knots. Since the unknotting number of \( K \) is less than or equal to \( u \), by Lemma 1.4,

3. \( z(\tilde{e}_1 \tilde{\gamma}_1 + \ldots + \tilde{e}_p \tilde{\gamma}_p) \) is represented by a 2-sphere in \( p \ CP^2 \# u(\overline{CP^2} \# \overline{CP^2}) \).

In case that \( z \) is even. By Lemma 1.3, (1) and (3),

\[
\left| \frac{8x - pz^2}{2} + p \right| \leq p + 2,
\]

\[
\left| \frac{-pz^2}{2} + p \right| \leq p + 2u.
\]
It follows that

\[
\frac{8x-4}{p} \leq z^2 \leq \frac{4u}{p} + 4.
\]

In case that \( z \) is odd and \( |z| \geq 3 \). By Lemma 1.3 and (3), there exists an odd prime integer \( q \) such that

\[
\left| \frac{-p z^2 (q^2 - 1)}{2q^2} + p \right| \leq p + 2u.
\]

This implies

(1-1) \quad z^2 \leq \frac{9}{2} \left( \frac{u}{p} + 1 \right).

We note that

(1-2) \quad 1 < \frac{9}{2} \left( \frac{u}{p} + 1 \right).

The inequations (1-1) and (1-2) imply that any odd integer \( z \) satisfies

(1-3) \quad 1 \leq z^2 \leq \frac{9}{2} \left( \frac{u}{p} + 1 \right).

Moreover if \( z \) is odd then \((2x + 1) \gamma + (2x - 1) \tilde{\gamma} + z (\tilde{\gamma}_1 + \ldots + \tilde{\gamma}_p)\) is a characteristic element in \( H_2(\mathbb{C}P^2 \# (p+1) \mathbb{C}P^2) \). By Lemmas 1.5, 1.6, 1.7 and (2),

(1-4) \quad 8x - z^2 = -1 \text{ if } p = 1,

(1-5) \quad 8x - 2z^2 = -2 \text{ if } p = 2,

(1-6) \quad 8x - pz^2 \leq -2 \text{ if } p \geq 3.

By (1-3), (1-4), (1-5) and (1-6), we have

\[
\begin{align*}
z^2 &= 8x + 1 \text{ if } p = 1, \\
z^2 &= 4x + 1 \text{ if } p = 2,
\end{align*}
\]
This completes the proof. □

Suppose that knots $K_+$ and $K_-$ have representatives in $S^3$ that are identical outside a 3-ball within which they are as in Figure 5. Then we say that $K_-$ is obtained from $K_+$ by changing a positive crossing and that $K_+$ is obtained from $K_-$ by changing a negative crossing. We define the **positive unknotting number** (resp. **negative unknotting number**) of a knot $K$, to be the minimum, over all sequences transforming $K$ to be a trivial knot, of the number of positive (resp. negative) crossings which are changed. If $K$ cannot be a trivial knot by changing only positive (resp. negative) crossings, then we define the positive unknotting number (resp. negative unknotting number) of $K$ is **infinite**.

**Figure 5**

1.9. **Lemma.** (Weintraub [18]) Let $K$ be a knot. If the positive unknotting number (resp. negative unknotting number) of $K$ is less than or equal to $u$, then there exists an embedded 2-disk $\Delta$ in $uCP^2 - \text{Int } B^4$ (resp. $uCP^2 - \text{Int } B^4$) such that $\Delta$ represents the zero element in $H_2(uCP^2 - \text{Int } B^4, \partial)$ (resp. $H_2(uCP^2 - \text{Int } B^4, \partial)$) and $\partial \Delta \subset \partial (uCP^2 - \text{Int } B^4)$ (resp. $\partial \Delta \subset \partial (uCP^2 - \text{Int } B^4)$) is $-K^1$.

1.10. **Lemma.** (Kervaire and Milnor [6]) Let $M$ be a connected, simply connected, closed 4-manifold. Let $\xi \in H_3(M)$ be a characteristic element. If $\xi$ is represented by an embedded 2-sphere in $M$, then $\xi^2 \equiv \sigma(M)$ mod 16.
1.11. **Lemma.** Let $p$ be a positive integer and $x$ a nonnegative integer. Let $K \in \mathcal{T}_\infty$ be a knot such that the negative unknotting number of $K$ is finite. If $K \in e$-Slice $(p\mathbb{CP}^2)$ then there exists an integer $z$ such that $z$ satisfies a condition

(a) $z^2 \leq 4 + \frac{4 - 8x}{p}$ and $z$ is even, or

\[
\begin{cases}
z^2 = 1 & \text{only if } x = 0 \text{ and } p = 1, 2, \\
z^2 = 1 & \text{only if } x \equiv 0 \mod 2 \text{ and } p \geq 3.
\end{cases}
\]

**Proof.** Suppose $K \in \mathcal{T}_\infty \cap e$-Slice ($p\mathbb{CP}^2$) and the negative unknotting number of $K$ is $u$. Since $K \in \mathcal{T}_\infty \cap e$-Slice ($p\mathbb{CP}^2$), there exists an integer $z$ such that

(4) $2\alpha - 2x\beta + z(\ell_1, \ell_1, + \ldots + \ell_p, \gamma_p) \in H_2(S^2 \times S^2 \# p\mathbb{CP}^2)$ is represented by a 2-sphere in $S^2 \times S^2 \# p\mathbb{CP}^2$ and

(5) $(2x - 1) \gamma + (2x + 1) \gamma + z(\ell_1, \ell_1, + \ldots + \ell_p, \gamma_p) \in H_2(\mathbb{CP}^2 \# (p + 1)\mathbb{CP}^2)$ is represented by a 2-sphere in $\mathbb{CP}^2 \# (p + 1)\mathbb{CP}^2$,

by Lemmas 1.1, 1.2 and the definition of evenly slice knots. Since the negative unknotting number of $K$ is $u$, by Lemma 1.9,

(6) $z(\ell_1, \gamma_1, + \ldots + \ell_p, \gamma_p)$ is represented by a 2-sphere in $p\mathbb{CP}^2 \# u\mathbb{CP}^2$.

In case that $z$ is even. By Lemma 1.3 and (4),

$$\left| -8x - pz^2 \right| + p \leq p + 2.$$ 

This implies

$$z^2 \leq 4 + \frac{4 - 8x}{p}.$$ 

In case that $z$ is odd. If $|z| \geq 3$, then by Lemma 1.3 and (6), there exists an odd prime integer $q$ such that

$$\left| -pz^2 (q^2 - 1) \right| + p - u \leq p + u.$$
It follows that
\[ z^2 \leq \frac{9}{2}. \]

This is a contradiction. Thus \(|z| = 1\). Moreover, by Lemmas 1.5, 1.7, 1.10 and (5), we have
\[ -8x - pz^2 = -p \quad \text{if} \quad p = 1, 2, \]
\[ -8x - pz^2 \equiv -p \mod 16. \]

Since \(|z| = 1\),
\[ -8x = 0 \quad \text{if} \quad p = 1, 2, \]
\[ -8x \equiv 0 \mod 16. \]

This implies
\[ x = 0 \quad \text{if} \quad p = 1, 2, \]
\[ x \equiv 0 \mod 2. \]

This completes the proof. \(\square\)

2. SLICE KNOTS IN \(CP^2\) or \(\overline{CP^2}\)

In this section we shall prove the following two theorems.

2.1. Theorem. Let \(x\) be a positive integer.

(a) If \(\text{Slice}(CP^2)\) contains \(T(2,4x-1)\), then \(2x-1, 2x\) or \(8x+1\) is a square number.

(b) If \(\text{Slice}(CP^2)\) contains \(T(2,4x+1)\), then \(2x, 2x+1\) or \(8x+1\) is a square number.

2.2. Theorem. Let \(t\) be a nonnegative integer. The set \(\overline{\text{Slice}(CP^2)}\) does not contain \(T(-2, 2t+1)\) if and only if \(t \geq 2\).

2.3. Remark. Since \(\text{Slice}(CP^2)\) contains a knot \(K\) if and only if \(\text{Slice}(CP^2)\) contains \(-K\). \(\text{Slice}(CP^2)\) contains \(T(l,m)\) if and only if
Slice \((CP^3)\) contains \(T(-l, m)\). It follows that Theorems 2.1 and 2.2 imply that there exist infinitely many integer \(x_i (i=1, 2, \ldots)\) such that \(T(2, 2x_i + 1)\) belongs to neither Slice \((CP^3)\) nor Slice \((CP^5)\) for any \(x_i\).

2.4. **Lemma.** For any \(T(2c, 4x+1)\) \((c=\pm 1, x\geq 0)\), there exists an embedded 2-disk \(\Delta\) in \(CP^2\# CP^3 - \text{Int} B^4\) such that \(\Delta\) represents an element \((2x+1+c)\gamma + (2x+1-c)\tilde{\gamma}\) in \(H_2(CP^2\# CP^3 - \text{Int} B^4, \partial)\) and \(\partial\Delta \subset \partial (CP^2\# CP^3 - \text{Int} B^4)\) is \(T(-2c, 4x+1)\).

**Proof.** By considering the Kirby's calculus as in Figure 2, we note that there exist mutually disjoint \(2x+2\) properly embedded 2-disk \(\Delta_1, \ldots, \Delta_{2x+2}\) in \(CP^2\# CP^3 - \text{Int} B^4\) such that \(\bigcup \Delta_i\) represents an element \((2x+2)\gamma + 2x\tilde{\gamma}\) in \(H_2(CP^2\# CP^3 - \text{Int} B^4, \partial)\) and \(\partial (\bigcup \Delta_i) \subset \partial (CP^2\# CP^3 - \text{Int} B^4)\) is as Figure 6. Since a \((-2, 4x+2)\)-torus link is obtained from \(\partial (\bigcup \Delta_i)\) by \(2x\)-fusion, there exist \(2x+1\) strips \(b_1, \ldots, b_{2x+1}\) connecting the link \(\partial (\bigcup \Delta_i)\) such that \(\Delta = \Delta_1 \cup \ldots \cup \Delta_{2x+2} \cup b_1 \cup \ldots \cup b_{2x+1}\) is an embedded 2-disk in \(CP^2\# CP^3 - \text{Int} B^4\) and \(\partial\Delta \subset \partial (CP^2\# CP^3 - \text{Int} B^4)\) is \(T(-2, 4x+1)\).

By considering the Kirby's calculus as in Figure 4, the above argument remains valid for \(T(-2, 4x+1)\).
Proof of Theorem 2.1. Suppose \( T(2, 4x - 1) \in \text{Slice}(\overline{CP^2}) \). Since the unknotting number of \( T(2, 4x - 1) \) is \( 2x - 1 \), \( T(2, 4x - 1) \in \mathcal{S}_x \) and \( \epsilon\text{-Slice}(\overline{CP^2}) = \text{Slice}(\overline{CP^2}) \), by Lemma 1.8, there exists an integer \( z \) such that \( z \) satisfies a condition

\[
(2-7) \quad 8x - 4 \leq z^2 \leq 8x \text{ and } z \text{ is even, or}
\]

\[
(2-8) \quad z^2 = 8x + 1.
\]

We set \( z = 2k \) in (2-7), then we have

\[
2x - 1 \leq k^2 \leq 2x.
\]

It follows that

\[
(2-9) \quad k^2 = 2x - 1, 2x.
\]

By (2-8) and (2-9), we obtain Theorem 2.1 (a).

Suppose \( T(2, 4x + 1) \in \text{Slice}(\overline{CP^2}) \). Since the unknotting number of \( T(2, 4x + 1) \) is \( 2x \) and \( T(2, 4x + 1) \in \mathcal{S}_x \), by Lemma 1.8, there exists an integer \( z \) such that \( z \) satisfies a condition

\[
(2-10) \quad 8x - 4 \leq z^2 \leq 8x + 4 \text{ and } z \text{ is even, or}
\]

\[
(2-11) \quad z^2 = 8x + 1.
\]

The fact that \( T(2, 4x + 1) \) belongs to \( \text{Slice}(\overline{CP^2}) \) and Lemma 2.4 imply that \( (2x + 2) \gamma + 2x \tilde{\gamma} + x \tilde{\gamma}_1 \in H_2(\overline{CP^2} \# 2CP^2) \) is represented by a 2-sphere in \( CP^2 \# 2CP^2 \). If \( z \) is even, then by Lemma 1.3, we have

\[
\left| \frac{8x + 4 - z^2}{2} + 1 \right| \leq 3.
\]

This implies

\[
(2-12) \quad 8x \leq z^2 \leq 8x + 12.
\]

By (2-10) and (2-12), we have

\[
(2-13) \quad 8x \leq z^2 \leq 8x + 4 \text{ and } z \text{ is even.}
\]

We set \( z = 2k \) in (2-13) then

\[
2x \leq k^2 \leq 2x + 1.
\]
It follows that

\[(2-14) \quad k^2 = 2x, \ 2x + 1.\]

By \((2-11)\) and \((2-14)\), we obtain Theorem 2.1 (b). \(\square\)

2.5. Proposition. If \(t \geq 3\) then \(\text{Slice}(\mathbb{C}P^2)\) does not contain \(T(-2, 2t+1)\).

Proof. Note that \(\mathcal{F}_x\) contains both \(T(-2, 4x-1)\) and \(T(-2, 4x+1)\) and that the negative unknotting number of \(T(-2, 4x-1)\) and that the negative unknotting number of \(T(-2, 4x+1)\) are finite. If \(\text{Slice}(\mathbb{C}P^2)\) contains \(T(-2, 4x-1)\) or \(T(-2, 4x+1)\), then by Lemma 1.11, there exists an integer \(z\) such that \(z\) satisfies a condition

\[(2-15) \quad z^2 = 8 - 8x \text{ and } z \text{ is even, or} \]
\[(2-16) \quad z^2 = 1 \text{ and } x = 0.\]

The conditions \((2-15)\) and \((2-16)\) imply

\[x = 0, 1.\]

This completes the proof. \(\square\)

2.5.1. Remark. By the proofs of Lemma 1.11 and Proposition 2.5, we note that if \(\text{Slice}(\mathbb{C}P^2)\) contains \(T(-2, 5)\) then there exists a properly embedded 2-disk \(\Delta\) in \(\mathbb{C}P^2 - \text{Int } B^4\) such that \(\Delta\) represents the zero element in \(H_2(\mathbb{C}P^2 - \text{Int } B^4, \partial)\) and \(\partial \Delta \subset \partial(\mathbb{C}P^2 - \text{Int } B^4)\) is \(T(-2, 5)\).

2.6. Proposition. The set \(\text{Slice}(\overline{\mathbb{C}P^2})\) does not contain \(T(-2, 5)\).

Proof. Suppose \(\text{Slice}(\overline{\mathbb{C}P^2})\) contains \(T(-2, 5)\). Remark 2.5.1 and Lemma 2.4 imply that \(2\gamma + 4\gamma \in H_2(\mathbb{C}P^3 \# \overline{\mathbb{C}P^2})\) is represented by a 2-sphere in \(\mathbb{C}P^2 \# 2\overline{\mathbb{C}P^2}\). By Lemma 1.3, we have

\[\left| \frac{4 - 16}{2} + 1 \right| \leq 3.\]

This is a contradiction. \(\square\)
3. EVENLY SLICE KNOTS IN $n_1 CP^2 \# n_2 \overline{CP}^2$

In [15], Norman proved that $\text{Slice}(CP^2 \# CP^2)$ is equal to the set of knots, but the following theorem implies that there exist infinitely many knots that do not belong to $e$-$\text{Slice}(CP^2 \# CP^2)$, i.e., $e$-$\text{Slice}(CP^2 \# CP^2)$ is a proper subset of $\text{Slice}(CP^2 \# CP^2)$.

3.1. Theorem. Let $t$ be a nonnegative integer and $e = \pm 1$. The set $e$-$\text{Slice}(CP^2 \# CP^2)$ contains $T(2e, 2t + 1)$ if and only if $t = 0$ or $1$.

3.2. Lemma. (Hirai [4]) Let $\xi \in H_5(2(CP^2 \# \overline{CP}^2))$ be a characteristic element. The element $\xi$ represented by a 2-sphere in $2(CP^2 \# \overline{CP}^2)$ if and only if $\xi^2 = 0$.

3.3. Proposition. For $e = \pm 1$, if $t \geq 3$ then $e$-$\text{Slice}(CP^2 \# \overline{CP}^2)$ does not contain $T(2e, 2t + 1)$.

Proof. Let $x$ be a nonnegative integer. If either $T(2e, 4x - 1)$ or $T(2e, 4x + 1)$ belongs to $e$-$\text{Slice}(CP^2 \# CP^2)$ then there exists an integer $z$ such that

$$2\alpha + 2ex\beta + z(e_1 \gamma_1 + \delta_1 \gamma_1) \in H_2(S^2 \times S^2 \# CP^2 \# \overline{CP}^2)$$ is represented by a 2-sphere in $S^2 \times S^2 \# CP^2 \# \overline{CP}^2$ and

$$2(2\alpha + e) \gamma + (2x - e) \gamma + z(e_1 \gamma_1 + \delta_1 \gamma_1) \in H_5(2(CP^2 \# \overline{CP}^2))$$ is represented by a 2-sphere in $2(CP^2 \# \overline{CP}^2)$,

by Lemmas 1.1, 1.2 and the definition of evenly slice knots. If $z$ is even, then by Lemma 1.3 and (7),

$$\left| \frac{8ex}{2} \right| \leq 4.$$

This implies

$$x = 0, 1.$$
If $z$ is odd, then by Lemma 3.2 and (8),

$$8\varepsilon x = 0.$$ 

It follows that if $x \geq 2$, then neither $T(2\varepsilon, 4x-1)$ nor $T(2\varepsilon, 4x+1)$ belongs to $\varepsilon$-Slice $(CP^2 \# CP^2)$. This completes the proof. \(\square\)

3.4. Proposition. The set $\varepsilon$-Slice $(CP^2 \# CP^2)$ does not contain $T(2\varepsilon, 5)$ for $\varepsilon = \pm 1$.

Proof. Suppose $\varepsilon$-Slice $(CP^2 \# CP^2)$ contains $T(2\varepsilon, 5)$. Proof of Proposition 3.3 and Lemma 2.4 implies that there exists an even integer $z$ such that $(3+\varepsilon)\gamma + (3-\varepsilon)\gamma + z(\varepsilon_1 \gamma_1 + \bar{\varepsilon}_1 \gamma_1) \in H_2(2(CP^2 \# CP^2))$ is represented by a 2-sphere in $2(CP^2 \# CP^2)$. By Lemma 1.3, we have

$$\left| \frac{12\varepsilon}{2} \right| \leq 4.$$ 

This is a contradiction. \(\square\)

Proof of Theorem 3.1. By Propositions 3.3 and 3.4, if $i \geq 2$ then $\varepsilon$-Slice $(CP^2 \# CP^2)$ does not contain $T(2\varepsilon, 2i+1)$. If $i = 0$ or 1 then $\varepsilon$-Slice $(CP^2 \# CP^2)$ contains $T(2\varepsilon, 2i+1)$, see Proposition 3.7. \(\square\)

The same arguments as proof of Theorem 2.1 and Proposition 2.5 lead to the following Theorem 3.5 and Proposition 3.6, respectively.

3.5. Theorem. Let $x$ be a positive integer.

(a) If $\varepsilon$-Slice $(CP^2 \# CP^2)$ contains $T(2, 4x-1)$ then $x$ or $4x+1$ is a square number.

(b) If $\varepsilon$-Slice $(CP^2 \# CP^2)$ contains $T(2, 4x+1)$ then $x$, $x+1$ or $4x+1$ is a square number.

3.6. Proposition. If $i \geq 3$ then $\varepsilon$-Slice $(CP^2 \# CP^2)$ does not contain $T(-2, 2i+1)$.

3.7. Proposition. Let $K$ be a knot. If the positive unknotting number or the negative unknotting number of $K$ is less than or equal to $p$, then both $\varepsilon$-Slice $(pCP^2)$ and $\varepsilon$-Slice $(pCP^2)$ contain $K$. 

Proof. Suppose $K$ is a knot and the positive or negative unknotting number of $K$ is less than or equal to $p$. Let $L_e$ be the Hopf link in $\partial (\text{CP}^2 - \text{Int} B^4)$ with linking number $e (e = \pm 1)$. It is easily seen that $L_e$ bounds a properly embedded 2-disk in $\text{CP}^2 - \text{Int} B^4$ that represents an element $(1-e) \gamma$ in $H_2(\text{CP}^2 - \text{Int} B^4, \partial)$. Since the positive or negative unknotting number of $K$ is less than or equal to $p$, $K$ is obtained from $p$ copies of $L_e$ by complete fusion. It follows that $K$ bounds a properly embedded 2-disk in $p\text{CP}^2 - \text{Int} B^4$ that represents an element $(1-e) \sum_{i=1}^{p} e_i \gamma_i$ in $H_2(p\text{CP}^2 - \text{Int} B^4, \partial)$. This implies that $K$ belongs to $e\text{-Slice}(p\text{CP}^2)$.

The above argument remains valid to show that $K$ belongs to $e\text{-Slice}(p\text{CP}^2)$. This completes the proof. □

By Propositions 3.6 and 3.7, we have the following theorem.

3.8. Theorem. Let $t$ be a nonnegative integer. The set $e\text{-Slice}(2\text{CP}^2)$ does not contain $T(-2, 2t+1)$ if and only if $t \geq 3$.

3.9. Theorem. For any integer $p \geq 3$, $e\text{-Slice}(p\text{CP}^2)$ contains neither $T(2, 8p + 3)$ nor $T(-2, 8p + 3)$.

Proof. Suppose that $e\text{-Slice}(p\text{CP}^2)$ contains $T(2, 8p + 3)$. Since $T(2, 8p + 3)$ belongs to $\mathcal{S}_{2p+1}$ and the unknotting number of $T(2, 8p + 3)$ is $4p + 1$, by Lemma 1.8, there exists an integer $z$ such that $z$ satisfies a condition

\begin{align}
(3-17) & \frac{16p + 4}{p} \leq z^2 \leq \frac{16p + 4}{p} + 4 \text{ and } z \text{ is even}, \\
(3-18) & \frac{16p + 10}{p} \leq z^2 \leq \frac{9}{2} \left( \frac{4p + 1}{p} + 1 \right) \text{ and } z \text{ is odd}.
\end{align}

Since $p \geq 3$, (3-17) and (3-18) imply

$16 < z^2 < 25$ and $z$ is even,

$16 < z^2 < 25$ and $z$ is odd.

This is a contradiction.

Suppose that $e\text{-Slice}(p\text{CP}^2)$ contains $T(-2, 8p + 3)$. Since $T(-2, 8p + 3)$ belongs to $\mathcal{S}_{-2p-1}$ and the negative unknotting number of $T(-2, 8p + 3)$ is
finite, by Lemma 1.11, there exists an integer \( z \) such that \( z \) satisfies the following condition
\[
z^2 \leq 4 + \frac{-16p - 4}{p} < 0.
\]
This is a contradiction. \( \square \)

3.10. Claim. Let \( K \) be a knot. Neither \( e\text{-Slice}(pCP^2) \) nor \( e\text{-Slice}(p\overline{CP}^2) \) contains \( K \) if and only if \( e\text{-Slice}(p\overline{CP}^2) \) contains neither \( K \) nor \( -K' \).

3.11. Remark. By Theorem 3.9 and Claim 3.10, we have that \( T(2, 8p+3) \) belongs to neither \( e\text{-Slice}(pCP^2) \) nor \( e\text{-Slice}(p\overline{CP}^2) \) for any \( p \geq 3 \).

4. APPLICATIONS

4.1. Proposition. If \( K \in \mathcal{S}_p \), then \( K \) belongs to either \( e\text{-Slice}(pCP^2) \) or \( e\text{-Slice}(p\overline{CP}^2) \).

Proof. If \( K \in \mathcal{S}_p \), then there exists a 2-disk \( D^2 \) and a trivial knot \( K_0 \) in \( S^3 \) such that \( K \) is obtained from \( K_0 \) by \( \frac{2}{p} \)-Dehn surgery along \( \partial D^2 \). We take the parallel copies \( D_1^2, \ldots, D_p^2 \) of \( D^2 \) as in Figure 7. It is easily seen that \( K \) is obtained from \( K_0 \) by Dehn surgery along \( \partial (\bigcup D_i^2) \) in which the surgery coefficients are all \( \varepsilon \). Suppose that \( K_0 \) and \( \bigcup D_i^2 \) are in the boundary of a 4-ball \( B_0^4 \), then \( K_0 \) bounds a properly embedded 2-disk \( \Delta \) in \( B_0^4 \). Let \( \{h_i^2\} \) \((1 \leq i \leq p)\) be 2-handles on \( B_0^4 \) whose attaching sphere are \( \partial D_i^2 \) and all framings are \( \varepsilon \). We note that \( K_0 \subset \partial (B_0^4 \cup \bigcup h_i^2) \) is \( K \). \( K \) bounds the 2-disk \( \Delta \) in \( B_0^4 \cup \bigcup h_i^2 \) and \( B_0^4 \cup \bigcup h_i^2 \) is diffeomorphic to either punctured \( pCP^2 \) or punctured \( p\overline{CP}^2 \). Let the punctured \( pCP^2 \) and punctured \( p\overline{CP}^2 \) be denoted by \( pCP^2 - \text{Int} B^4 \) and \( p\overline{CP}^2 - \text{Int} B^4 \), respectively. Suppose the linking number \( lk(K_0, \partial D^2) = z \) then \( lk(K_0, \partial D^2) \) \((1 \leq i \leq p)\) are the same number as \( z \). It is not hard to see that \( \Delta \) represents either an element \( z(\varepsilon_1 \gamma_1 + \ldots + \varepsilon_p \gamma_p) \) in \( H_2(pCP^2 - \text{Int} B^4, \partial) \) or an element \( z(\tilde{\varepsilon}_1 \tilde{\gamma}_1 + \ldots + \tilde{\varepsilon}_p \tilde{\gamma}_p) \) in \( H_2(p\overline{CP}^2 - \text{Int} B^4, \partial) \). This implies that \( K \) belongs to either \( e\text{-Slice}(pCP^2) \) or \( e\text{-Slice}(p\overline{CP}^2) \). \( \square \)

By Remark 3.11, Proposition 4.1 and the definition of evenly slice knots, we have the following theorem.

4.2. Theorem. For any integer \( p \geq 3 \), \( \mathcal{S}_p \) does not contain any knot that is cobordant to \( T(2, 8p+3) \).
Figure 7
By Lemmas 1.8 and 1.11, we have the following proposition.

4.3. Proposition. For any $p$ ($1 \leq p \leq 5$), $e$-Slice($p\overline{CP}^2$) contains neither $T(2, 75)$ nor $T(-2, 75)$.

By Claim 3.10, Propositions 4.1, 4.3 and the definition of evenly slice knots, we have the following proposition.

4.4. Proposition. For any $p$ ($1 \leq p \leq 5$), $\mathcal{K}_p$ does not contain any knot that is cobordant to $T(2, 75)$.

4.5. Lemma. (Motegi [14]) If $p \geq 6$ then $\mathcal{K}_p$ does not contain any composite knot.

Let $K$ be a nontrivial slice knot. Proposition 4.4 and Lemma 4.5 imply that $\mathcal{K}_p$ does not contain $T(2, 75) \# K$ for any $p \geq 1$. Hence we have the following theorem that gives a negative answer to Question 2.

4.6. Theorem. There exist infinitely many knots that do not belong to any $\mathcal{K}_p$ ($p \geq 1$).

Let $K$ be a knot in $\partial(CP^2 \# \overline{CP}^2 - \text{Int} \ B^4)$. If $K$ is obtained from a slice knot by a $(\pm 1, \pm 1)$-twisting, then by proof of Proposition 4.1, $K$ bounds a properly embedded 2-disk in $CP^2 \# \overline{CP}^2 - \text{Int} \ B^4$ that represents an element $\pm \gamma_1$ or $\pm \gamma_1$ in $H_2(CP^2 \# CP^2 - \text{Int} \ B^4, \partial)$. It follows that $K$ bounds a properly embedded 2-disk in $CP^2 \# \overline{CP}^2 - \text{Int} \ B^4$ that represent an element $\pm \gamma_1 + \gamma_1$ or $\gamma_1 \pm \gamma_1$ in $H_2(CP^2 \# CP^2 - \text{Int} \ B^4, \partial)$. We have the following proposition.

4.7. Proposition. If $K$ is obtained from a slice knot by $(\pm 1, \pm 1)$-twisting, then $K$ belongs to $e$-Slice($CP^2 \# CP^3$).

Since a $(\pm 1, \pm 1)$-twisting does not change the Arf invariant of a knot, thus $T(2e, 3)$ cannot be obtained from a slice knot by a $(\pm 1, \pm 1)$-twisting. By Theorem 3.1, Proposition 4.7 and the definition of evenly slice knots, we have the following theorem.

4.8. Theorem. Let $t$ be a nonnegative integer and $e = \pm 1$. A knot cobordant to $T(2e, 2t+1)$ is obtained from a slice knot by a $(\pm 1, \pm 1)$-twisting if and only if $t = 0$. 
If $2r + 1 \equiv \pm 1 \text{ mod } 8$, then the Arf invariant of $T(2e, 2r+1)$ is zero (for example, see p266 in [5]). Thus Theorem 4.8 gives infinitely many counterexamples to Conjecture.

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References


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