Isometries and Automorphisms of the Spaces of Spinors

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ABSTRACT. The relationships between the JB*-triple structure of a complex spin factor $\mathcal{S}$ and the structure of the Hilbert space $\mathcal{H}$ associated to $\mathcal{S}$ are discussed. Every surjective linear isometry $L$ of $\mathcal{S}$ can be uniquely represented in the form $L(x) = \mu U(x)$ for some conjugation commuting unitary operator $U$ on $\mathcal{H}$ and some $\mu \in \mathbb{C}$, $|\mu| = 1$. Automorphisms of $\mathcal{S}$ are characterized as those linear maps (continuity not assumed) that preserve minimal tripotents in $\mathcal{S}$ and the orthogonality relations among them.

§0. INTRODUCTION

The spaces of spinors were introduced by E. Cartan in [1] to solve the problem of analytic classification of bounded symmetric domains in $\mathbb{C}^n$, and they also arise in the quantization of free fermionic fields [9, p. 104]. More recently, these spaces have been considered in various problems in the context of infinite dimensional holomorphy by Harris, Kaup and others. In this note, an arbitrary spinor space $\mathcal{S}$ is considered and the relationship between the «triple structure» of $\mathcal{S}$ and the structure of its underlying Hilbert space $\mathcal{H}$ is discussed. In §2, we prove that any surjective linear isometry $L$ of $\mathcal{S}$ can be represented in the form $L = \mu U$ for some $\mu \in \mathbb{C}$, $|\mu| = 1$ and some conjugation commuting surjective linear isometry of $\mathcal{H}$. Since surjective linear isometries of $\mathcal{S}$, and conjugation commuting unitary operators on $\mathcal{H}$, are the same as automorphisms of the corresponding structures of $\mathcal{S}$ and $\mathcal{H}$, our result can be rephrased by saying that, except for an automorphism, the spaces of spinors are uniquely determined by their underlying Hilbert space. This is a result that anyone could expect, though the authors have found no precise reference for the statement and proof. In §3, the sets $\text{Min}(\mathcal{S})$ and

1 Supported by the Xunta de Galicia, project XUGA 20702 B 90.
1991 Mathematics Subject Classification: 32M15, 47C10, 46G20.
Extr(\mathcal{S}) of minimal and maximal tripotents of \mathcal{S} are discussed. In §4 we prove that any linear mapping (continuity is not assumed) that preserves minimal tripotents and the orthogonality relations among them is an automorphism of \mathcal{S}, and that any holomorphic automorphism of the unit ball of \mathcal{S} is uniquely determined by its values at the set \text{Min}(\mathcal{S}) \cup \{0\}.

§1. NOTATION AND PRELIMINARY RESULTS

Let \( H \) be a complex Hilbert space with \( \dim(H) > 1 \). We recall ([2] p. 16, [4] p. 358) that a Cartan factor of type IV, also called a spinor or a spin factor, is a norm closed selfadjoint complex subspace \( \mathcal{S} \) of \( \mathcal{L}(H) \) such that \( a^2 a = a^*a = a^*a = 2(a|b)1_H \), where \( \mathcal{L}(H) \) is the \( C^* \)-algebra of bounded linear operators on \( H \) and \( 1_H \) is the identity operator. For \( a \) and \( b \) in \( \mathcal{S} \), there is a unique complex number \( (a|b) \) such that

\[
ab^* + b^*a = 2(a|b)1_H
\]

and the mapping \( (a, b) \to (a|b) \) defines an inner product on \( \mathcal{S} \) whose associated norm, denoted by \( \| . \| \), is equivalent to the usual operator norm \( \| . \|_{\infty} \) induced by \( \mathcal{L}(H) \) on \( \mathcal{S} \). Let us define

\[
\chi = \{ a \in \mathcal{S} | a = a^* \}.
\]

The norms \( \| . \| \) and \( \| . \|_{\infty} \) coincide on \( \chi \), and we have the topological direct sum decomposition

\[
\mathcal{S} = \chi \oplus i\chi
\]

In particular, \( \mathcal{S} \) is a complex Hilbert space with conjugation \( a \to a^* \), \( a \in \mathcal{S} \), the hilbertian norm and the operator norm being related by

\[
\|a\|_2^2 = \|a\|^2 + \|a\|^2 - \| (a|a^*) \|^2
\]

On the other hand, \( \mathcal{S} \) is a \( J^* \)-algebra of operators, i.e., \( \mathcal{S} \) is a norm closed complex subspace of \( \mathcal{L}(H) \) such that the triple product

\[
\{ab^*c\} = \frac{1}{2}(ab^*c + cb^*a)
\]

is in \( \mathcal{S} \) whenever \( a, b, \) and \( c \) are in \( \mathcal{S} \). The \( J^* \)-algebra structure and the Hilbert space structure are linked by the formula ([4] p. 358)

\[
\{aa^*a\} = 2(a|a)a - (a|a^*)a^* \quad (a \in \mathcal{S})
\]
An alternative introduction of the spaces of spinors is the following: Let $K$ be a complex Hilbert space with conjugation $\bar{}$ and inner product $\langle . | . \rangle$. Define a triple product by

$$\{x y z\} = \langle x | y \rangle z - \langle z | x \rangle y + (z \bar{y}) x.$$  

Then $K$ with the conjugation $\bar{}$, the triple product $\{ . . \}$ and the norm $\| . \|_\infty$ given by

$$\| x \|_\infty^2 = \| x \|^2 + \left[ \| x \|^4 - |\langle x | x \rangle|^2 \right]^{1/2}$$  

is a Cartan factor of type IV. The norms $\| . \|$ and $\| . \|_\infty$ are referred to as the Hilbert and the Lie norm on $K$, their unit balls being denoted by $B$ and $B_\infty$.

Some other basic facts on $J^*$-algebras are needed in the sequel. Let $L: \mathcal{S} \to \mathcal{S}$ be a vector space isomorphism (continuity not assumed) between the $J^*$-algebras $\mathcal{S}$ and $\mathcal{S}$. Then $L$ commutes with the triple product, i.e.,

$$L\{a b c\} = \{L(a) L(b) L(c)\} \quad (a, b, c \in \mathcal{S}),$$

if and only if $L$ is an isometry. In that case, $L$ is said to be a $J^*$-isomorphism. An element $a \in \mathcal{S}$ is said to be a tripotent if $a \neq 0$ and $\{a a^* a\} = a$, and in that case $\| a \|_\infty = 1$. The tripotent $a$ is said to be minimal if, to each $x \in \mathcal{S}$, there exists $\lambda_x \in \mathbb{C}$ such that

$$\{a x a\} = \lambda_x a.$$  

$J^*$-isomorphisms preserve tripotents and minimal tripotents. The set $\text{Min}(\mathcal{S})$ of minimal tripotents of $\mathcal{S}$ is given by ([3] p. 179)

$$\text{Min}(\mathcal{S}) = \{ a \in \mathcal{S} | a^2 = 0 \}.$$  

§2. ISOMETRIES OF THE SPACES OF SPINORS

In this section, $\mathcal{S}$ and $\chi$ stand for a fixed space of spinors and its selfadjoint part.

Lemma. Let $L: \mathcal{S} \to \mathcal{S}$ be any surjective linear $\| . \|_\infty$-isometry of $\mathcal{S}$. Then there exists a complex number $\lambda \in \mathbb{C}$, $|\lambda| = 1$, such that

$$L(a)^* = \lambda L(a) \quad (a \in \chi).$$  

(2.1)

As a consequence, $\| L a \| = \| a \|_\infty$ holds for all $a$ in $\chi$.  

Proof: Let \( a \in \chi, a \neq 0 \), be given. By (1.3)
\[
\{a a^* a\} = \|a\|^2 a.
\]
Thus applying \( L \)
\[
L\{a a^* a\} = \|a\|^2 L(a)
\]
whence again by (1.3)
\[
L\{a a^* a\} = \{L(a) L(a)^* L(a)\} = 2\|L(a)\|^2 L(a) - (L(a) | L(a)^*) L(a)^*
\]
If \((L(a) | L(a)^*) = 0\) then by (1.1) and (1.3)
\[
2 L(a)^2 = 2(L(a) | L(a)^*) L = 0
\]
i.e., \( L(a) \) is a minimal tripotent; but then \( a \) is also a minimal tripotent and so \( a^2 = 0 \) which, together with \( a \in \chi \), implies \( a = 0 \), a contradiction. From (2.2)
\[
L(a)^* = \lambda L(a) \quad (a \in \chi)
\]
where \( |\lambda| = 1 \) and
\[
\lambda = \frac{2 \|L(a)\|^2 - \|a\|^2}{(L(a) | L(a)^*)} \quad (a \in \chi).
\]
We claim that \( \lambda \) does not depend on \( a \in \chi \). Indeed, let \( b \in \chi \) be given so that \( a, b \) are linearly independent (this is possible since by assumption \( \dim \mathcal{S} > 1 \)). By (2.3) there are unitary numbers \( \lambda, \mu, \nu \in \mathbb{C} \), such that
\[
L(a) = \lambda L(a)^*, \quad L(b) = \mu L(b)^*, \quad L(a + b) = \nu [L(a + b)]^*
\]
whence by the linearity of \( L \) and the independence of \( L(a) \) and \( L(b) \), we get \( \lambda = \mu = \nu \). Using (2.3) and the expression of \( \lambda \),
\[
\|L(a)\|^2 = (L(a) | L(a)) = (L(a) | \lambda L(a)^*) = \lambda (L(a) | L(a)^*) = 2 \|L(a)\|^2 - \|a\|^2
\]
whence, by the coincidence of \( \| . \|_\infty \) and \( \| . \| \) on \( \chi \), we get
\[
\|L(a)\|^2 = \|a\|^2 - \|a\|^2_\infty
\]

Theorem. Let \( \mathcal{S} \) be any spin factor, and let \( L: \mathcal{S} \to \mathcal{S} \) be any surjective vector space isomorphism. Then the following statements are equivalent:

1. \( L \) is an \( \| . \|_\infty \)-isometry of \( \mathcal{S} \).
2. There is a unitary operator $U$ on the Hilbert space $\mathcal{S}$ with $U(a^*) = U(a)^*$ for all $a$ in $\mathcal{S}$, and there is a number $\mu \in \mathbb{C}$ with $|\mu| = 1$ such that

$$L(a) = \mu U(a) \quad (a \in \mathcal{S}).$$

In particular, any $\| \cdot \|$-isometry of $\mathcal{S}$ is an isometry for the underlying Hilbert space.

**Proof:** "$2 \Rightarrow 1$" is immediate, so we show "$1 \Rightarrow 2". By the previous lemma

$$L(a)^* = \lambda L(a) \quad (a \in \chi)$$

for some $\lambda \in \mathbb{C}$, $|\lambda| = 1$. Let $\mu \in \mathbb{C}$ be such that $\mu^2 = \lambda$, and define $U: \mathcal{S} \to \mathcal{S}$ by $U = : \mu L$. Then, for $a \in \chi$ we have

$$U(a) = \mu L(a) = \tilde{\mu} L(a)^* = [(\mu L)(a)]^* = U(a)^*$$

i.e., $U(\chi) \subseteq \chi$, and so

$$U(b^*) = [U(b)]^* \quad (b \in \mathcal{S}).$$

Since $L = \tilde{\mu} U$, in order to prove the theorem it suffices to show that $U$ is an isometry for the hilbertian norm $\| \cdot \|$ on $\mathcal{S}$. But this is a consequence of the sesquilinearity of the scalar product and the fact that $U$ is an $\| \cdot \|$-isometry on the selfadjoint part $\chi$ of $\mathcal{S}$.

Note. The authors thank Prof. Rodríguez Palacios for simplifying their original proof of this theorem.

§3. **EXTREME POINTS OF THE UNIT LIE BALL OF $\mathcal{S}$**

A tripotent $e$ of $\mathcal{S}$ is said to be regular if the operator $\xi \rightarrow \{e e^* \xi\}$, $\xi \in \mathcal{L}$, is invertible in $\mathcal{S} (\mathcal{L})$, and this occurs ([7] p. 190) if and only if $e$ is a (real or complex) extreme point of the unit Lie ball of $\mathcal{S}$, whose set is denoted by $\text{Extr}(\mathcal{S})$. We have ([9] p. 37)

**Theorem.** If $\mathcal{S}$ is any spin factor, then the following equalities hold:

$$\{ e \in \mathcal{S} \mid e \text{ is a unitary operator} \} = \{ e \in \mathcal{S} \mid e \text{ is a normal operator and } \|e\|_\infty = 1 \}$$

$$= \{ e \in \mathcal{S} \mid e \text{ is a normal operator and } \|e\| = 1 \} = \{ e \in \mathcal{S} \mid e \text{ is a real extreme point of } B \} = \{ e \in \mathcal{S} \mid \|e\|_\infty = 1 \text{ and } e^* = \lambda e, \lambda \in \mathbb{C}, |\lambda| = 1 \}.$$
Proof: Let us denote by $S_k$, $1 \leq k \leq 5$, the sets above. The inclusion $S_1 \subset S_2$ is clear, and $S_2 \subset S_3$ follows by (1.1). We now prove $S_3 \subset S_4$. Let $e \in S_3$ from (1.1), the assumption $\|e\| = 1$ and the normality of $e$ we get

\begin{equation}
    ee^* = 1_H = e^*e
\end{equation}

whence by composing with $e$, $[ee^* e] = e$ which shows that $e$ is a tripotent. From (3.1) $[ee^* x]1 = \frac{1}{2} (ee^* x + xe^* e) = x$ for $x \in \mathcal{S}$, which shows the regularity of $e$, hence $e \in \text{Extr}(\mathcal{S})$

We now prove "$S_4 \subset S_5$." Let $e \in S_4$, hence in particular

\begin{equation}
    e = [ee^* e] = ee^* e
\end{equation}

If here we compose with $e$ and use the fact that $e^2 = \alpha 1_H$ for some $\alpha \in \mathbb{C}$ (recall the definition of $\mathcal{S}$), we obtain $\alpha (ee^* - 1_H) = 0$. But $\alpha \neq 0$ as otherwise $e^2 = 0 = e^2*$ and so

\begin{equation}
    [ee^* e^*] = \frac{1}{2} (ee^{*2} + e^{*2} e) = 0
\end{equation}

which contradicts the regularity of $e$. Thus $ee^* = 1_H$ and multiplying by $e$ on the left, $ee^* = e$, where $|\alpha| = 1$ since $e \in \text{Extr}(\mathcal{S})$ entails $\|e\|_\infty = 1$. The inclusion $S_4 \subset S_3$ is trivial.

§4. BOUNDARY BEHAVIOUR OF AUTOMORPHISMS

In this section, $\mathcal{S}$ denotes an arbitrary $JB^*$-triple of finite rank ([8], p.5.4). $\|\cdot\|_\infty$ denotes its unique $JB^*$-norm, $B_\infty$ is the unit ball of $\mathcal{S}$, and $\mathcal{G} = \text{Aut}(B_\infty)$ is the group of all holomorphic automorphisms of $B_\infty$. It is known ([5], prop. 3.2) that each $g \in \mathcal{G}$ extends to a holomorphic mapping on a neighbourhood of $\partial B_\infty$, and that $g$ maps the boundary $\partial B_\infty$ of $B_\infty$ onto itself. It is also known that ([8], p.3.10) that each $x \in \mathcal{S}$, $x \neq 0$, admits a spectral representation of the form

\begin{equation}
    x = \sum_{k=1}^{n} \lambda_k e_k
\end{equation}

for some pairwise orthogonal minimal tripotents $e_k$ and some uniquely determined scalars $\lambda_k$, $1 \leq k \leq n$ with

\begin{align}
    \lambda_1 \geq \lambda_2 \geq \ldots \geq 0, \\
    \|x\|_\infty = \max \{ \lambda_k \mid 1 \leq k \leq n \}
\end{align}
Theorem. Let $\mathcal{S}$ be any finite rank JB*-triple, and let $L: \mathcal{S} \rightarrow \mathcal{S}$ be any linear mapping (continuity not assumed). Then the following statements are equivalent:

1. $L$ is a $J^*$-automorphism of $\mathcal{S}$.

2. $L$ maps $\text{Min}(\mathcal{S})$ onto itself and preserves the orthogonality relations on $\text{Min}(\mathcal{S})$,

\[
L[\text{Min}(\mathcal{S})] = \text{Min}(\mathcal{S}) \quad [a, b \in \text{Min}(\mathcal{S}), \ a \perp b \Rightarrow L(a) \perp L(b)].
\]

Proof: We show that "2 $\Rightarrow$ 1" as the converse is trivial. Let $x \in \mathcal{S}$ and let

\[(4.1) \ x = \sum_{k=1}^{n} \lambda_k L(e_k)
\]

where by (4.3), $L(e_k)$, $1 \leq k \leq n$, are pairwise orthogonal minimal tripotents and so, by the properties of the spectral representation and (4.2),

\[
\|L(x)\|_{\infty} = \|x\|_{\infty}.
\]

Besides, $L$ is surjective. Indeed, if

\[
y = \sum_{1}^{m} \mu_j e_j \in \mathcal{S},
\]

by (4.3) there are tripotents $f_j$, $1 \leq j \leq m$ (orthogonality is not needed now), such that $L(f_j) = e_j$, hence $x = \sum_{1}^{m} \mu_j f_j$ satisfies $L(x) = y$. Thus $L$ is a $J^*$-automorphism.

Corollary. Any $J^*$-automorphism $L$ of a finite rank JB*-triple $\mathcal{S}$ is uniquely determined by its values at the set $\text{Min}(\mathcal{S})$.

Corollary. A holomorphic automorphism of the unit ball $B_n$ of a finite rank JB*-triple is uniquely determined by its values at the set $\{0\} \cup \text{Min}(\mathcal{S})$.

Proof: Let $f$ and $g$ in $\text{Aut}(B_n)$ be such that $f(0) = g(0) = a$ and $f(e) = g(e)$ for all $e \in \text{Min}(\mathcal{S})$. Take any $h \in \text{Aut}(B_n)$ such that $h(a) = 0$. Then $L = (hg)^{-1} * (hf)$ fixes the origin, hence by Cartan's uniqueness theorem, $L$ is linear. Since $L$ fixes any minimal tripotent of $\mathcal{S}$, we have $L = \text{Id}_{\mathcal{S}}$ and $f = g$. 
References