On convolution operator in Orlicz spaces

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ABSTRACT. The Orlicz spaces defined on a locally compact abelian group are considered. The main results consist in presenting sufficient and necessary conditions, expressed in terms of generated functions \( \varphi \), for embeddings \( L_{q_1} \hookrightarrow L_{q_2}, E_{q_1} \hookrightarrow E_{q_2}, E_{q_1} \hookrightarrow L_{q_2} \text{ and } L_{q_1} \hookrightarrow E_{q_2} \), where \( L_{q_1} \) are Orlicz spaces and \( E_{q_1} \) are their subspaces consisting of all order continuous elements. Some results of the paper are an extension and generalization of those contained in [2], [3], [8] and [10].

INTRODUCTION

The Young theorem ([2], [8]) including sufficient conditions for \( L \hookrightarrow L \hookrightarrow L(1 < p, q, r < \infty) \) has been known for many years. In [8], O'Neil generalizes this result to Orlicz spaces stating sufficient condition for \( L_{q_1} \hookrightarrow L_{q_2} \). From the other hand there are known sufficient and necessary conditions for the space \( L(1 < p < \infty) \) to be a Banach algebra under convolution as multiplication [10]. A generalization of this result for the Orlicz space is included in [3]. Our main topic consists in finding necessary and sufficient conditions for embeddings of \( L_{q_1} \hookrightarrow L_{q_2} \) and \( E_{q_1} \hookrightarrow E_{q_2} \). We investigate also the other embeddings like \( E_{q_1} \hookrightarrow L_{q_2} \) and \( L_{q_1} \hookrightarrow E_{q_2} \). The Young theorem, the O'Neill's results and the results concerning the Lebesgue and Orlicz spaces as Banach algebras are obtained as corollaries of our results. In particular we get the necessity of the Young theorem, which seems to be not known so far. We also get the answer to the problem given by B. Gramsch in [1].

The first part is devoted to general modular spaces. We give some equivalent conditions in order to a bilinear operator defined on a Cartesian product of modular spaces \( X_{q_1} \times X_{q_2} \) act to another modular space \( X_{q_3} \). The results of this part are applied to the second one, where the Orlicz spaces and convolution are investigated as modular spaces and the bilinear operator, respectively. The important role is played by conditions (+) and (++) expressing some connections between Young functions \( \varphi \). There are a few versions of those

Editorial de la Universidad Complutense. Madrid, 1989
conditions depending on the kind of a group $G$ and the Haar measure $\mu$. In Theorems 8 and 9 there are given sufficient conditions for $L_\ast \ast L_\ast \ast E_\ast \ast E_\ast$ by means of the condition $(\ast)$ and $(\ast\ast)$. For a discrete group it is possible to prove a converse statement (Theorem 10) without any additional assumptions on the group $G$, whereas for a nondiscrete group the full converse statement (Theorems 11, 14) is obtained under the assumption of the so called condition $(\ast)$ on the group $G$. It is not difficult to check that the groups like $(\mathbb{R}, +), (\mathbb{K}, +), (\mathbb{T}, \cdot), (\mathbb{R}\setminus \{0\}, \cdot), (\mathbb{K}\setminus \{0\}, \cdot)$ satisfy condition $(\ast)$ (Remark 12). At the end there are a number of corollaries including among others, the Young theorem with necessary and sufficient conditions for a large class of locally compact abelian groups.

1.

Let us now agree on some terminology. Let $\mathbb{R}, \mathbb{K}, \mathbb{N}$ stand for real, complex and natural numbers respectively. Let $X$ be a complex or real vector space. Recall some notions connected with modular spaces [7]. A functional $p : X \to [0, +\infty]$ is called a convex modular if it satisfies the conditions

\begin{enumerate}[(1)]
    \item $p(0) = 0$; \(\forall x, p(\lambda x) = \lambda p(x)\) for all $\lambda \in \mathbb{R}$ (in the real case),
    \item $p(ax + \beta y) \leq \alpha p(x) + \beta p(y)$ if $a, \beta \geq 0$ and $a + \beta = 1$.
\end{enumerate}

For any convex modular define the space $X_p = \{x \in X : \lim_{\lambda \to 0} p(\lambda x) = 0\} = \{x \in X : p(x) < \infty\}$ for some $\lambda > 0$] called a modular space and $X_p^\ast = \{x \in X : p(\lambda x) < \infty\}$ for all $\lambda > 0]$ a subspace of $X_p$ called the subspace of finite elements. The functional $\|x\|_p = \inf\{\varepsilon > 0 : p(x/\varepsilon) \leq 1\}$, $x \in X_p$ is a norm in $X_p$. The subspace $X_p^\ast$ considered with the same norm is closed in $X_p$.

1.1. Theorem. Let $p(i=1,2,3)$ be modulars defined on $X$ and $\gamma : X_{p_1} \times X_{p_2} \to X$ be a bilinear operator. The following conditions are equivalent

\begin{enumerate}[(i)]
    \item For every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x \in X_{p_1}, y \in X_{p_2}$ if $p_i(x) \leq \delta$ and $p_i(y) \leq \delta$ then $p_i(\gamma(x,y)) \leq \varepsilon$.
    \item There exists $k_i > 0$ $(i=1,2,3)$ such that for $x \in X_{p_1}, y \in X_{p_2}$ if $p_i(x) \leq k_i$ and $p_i(y) \leq k_i$ then $p_i(\gamma(x,y)) \leq k_i$.
    \item There exists $k > 0$ such that if $x \in X_{p_1}, y \in X_{p_2}$ and $p_i(x) \leq 1$ and $p_i(y) \leq 1$ then $p_i(\gamma(x,y)) \leq 1$.
    \item There exists $c > 0$ such that $\|\gamma(x,y)\|_{p_3} \leq c\|x\|_{p_1}\|y\|_{p_2}$ for all $x \in X_{p_1}, y \in X_{p_2}$.
    \item $\gamma : X_{p_1} \times X_{p_2} \to X$ and $\gamma$ is continuous.
\end{enumerate}

If $\gamma : X_{p_1}^\ast \times X_{p_2}^\ast \to X$ then the analogous conditions to the above in which the spaces $X_{p_i}$ are replaced by $X_{p_i}^\ast (i=1,2)$ are equivalent.
Proof. It is enough to give a proof only for \( \gamma : X_1 \times X_2 \to X \). The implications (i)\( \to \) (ii) and (iv)\( \to \) (v) are evident and (iii)\( \to \) (iv) results directly from the definition of a norm in a modular space.

(ii)\( \to \) (iii) if \( \rho(x) \leq 1 \) and \( \rho(y) \leq 1 \) then
\[
\rho(\min(1,k)\cdot x) \leq k \quad \text{and} \quad \rho(\min(1,k)\cdot y) \leq k.
\]
So
\[
\rho(k\cdot \min(1,k)\cdot \gamma(x,y)) \leq \max(1,k).
\]
Then, by convexity of \( \varphi \),
\[
\rho(\gamma(x,y)) \leq 1 \quad \text{under} \quad k = k\cdot \min(1,k)/ \max(1,k).
\]
(v)\( \to \) (i) Suppose \( \gamma \) takes its values in \( X_1 \) and (i) is not satisfied. So there exist \( \varepsilon > 0 \) and sequences \( (x_n) \subset X_1, (y_n) \subset X_2 \) such that \( \rho(x_n) \leq 1/n \), \( \rho(y_n) \leq 1/n \) and \( \rho((1/n)\gamma(x_n,y_n)) = \varepsilon \). Taking \( \bar{x}_n = (1/n)x_n \), \( \bar{y}_n = (1/n)y_n \) we have \( \rho((\lambda\bar{x}_n) \leq \rho(\lambda/n\bar{x}_n) \leq 1/n \) and \( \rho((\lambda\bar{y}_n) \to 0 \), which implies that \( \|\bar{x}_n\|_p \to 0 \) and \( \|\bar{y}_n\|_p \to 0 \). However, \( \rho((\gamma(\bar{x}_n,\bar{y}_n)) = \rho((1/n)\gamma(x_n,y_n)) \geq \varepsilon \). Thus \( \|\gamma(\bar{x}_n,\bar{y}_n)\|_p \to 0 \) and \( \gamma \) is not continuous.

For some kinds of spaces \( X \), modulars \( \rho \) and operators \( \gamma \) one can show more.

1.2. Theorem. Let \( X \) be a vector lattice. Suppose \( \rho((|x_1|) \leq \rho((|x_2|) \) if \( |x_1| \leq |x_2| \). Moreover, let \( X_1 \) be complete and a bilinear operator \( \gamma : X_1 \times X_2 \to X \) be positive, i.e. \( \gamma(x,y) \geq 0 \) if \( x \geq 0 \) and \( y \geq 0 \) and let \( \gamma(x,y) \leq \gamma(|x|,|y|) \).

The following conditions are equivalent

1. \( \gamma : X_1 \times X_2 \to X_3 \) and \( \gamma \) is continuous.

2. \( \gamma : X_1 \times X_2 \to X_1 \).

3. There exists \( k > 0 \) such that for \( x \in X_1, y \in X_2 \) \text{ if } \rho(x) \leq 1 \text{ and } \rho(y) \leq 1 \text{ then } \rho(k\gamma(x,y)) \leq 1.

The above conditions in which \( X_i \) are replaced by \( X_i' \) for \( i = 1,2 \), are equivalent, too.

Proof. By the previous theorem only the implication 2.\( \to \) 3.\( \ast \) needs a proof.
We shall show it in the case of \( X_i' \) \( i=1,2 \). For a contrary, let 3.\( \ast \) be not fulfilled. So there exist sequences \( (x_n) \subset X_1', (y_n) \subset X_2' \) such that \( \rho(x_n) \leq 1 \), \( \rho(y_n) \leq 1 \) and \( \rho((1/2^n)\gamma(x_n,y_n)) \geq 1 \).

The elements \( z = \sum_{n=1}^{\infty} \frac{|x_n|}{2^n} \), \( w = \sum_{n=1}^{\infty} \frac{|y_n|}{2^n} \) belong to \( X_1', X_2' \), respectively,
because of convexity of \( p \), and the fact that \( X' \) are closed subspaces of \( X \).

However,

\[
p_i((1/k)\gamma(z,w)) \geq p_i((1/2^mk)\gamma(x\lambda,y\lambda)) \\
\geq p_i((1/2^mk)\gamma(x\lambda,y\lambda)) > l,
\]

by assumed properties of \( \gamma \). Thus \( \gamma(z,w) \notin X' \), which ends the proof.

The results of this section will be applied in the next one to Orlicz spaces and convolution as modular spaces and bilinear operator, respectively.

2.

Let \( \varphi : [0, +\infty) \to [0, +\infty] \) be convex, left-continuous not identical to zero and infinity, \( \varphi(0) = 0 \) and \( \varphi(+\infty) = +\infty \). In the sequel this function will be called a Young function. We say that a Young function is finite if it is finite on \([0, +\infty)\). A generalized inverse function \( \varphi^{-1} : [0, +\infty) \to [0, +\infty] \) is defined as

\[
\varphi^{-1}(y) = \inf \{ x \geq 0 : \varphi(x) > y \}, \text{ where } \inf \emptyset = +\infty.
\]

Let \( a, b \) be reserved for the following numbers

\[
a = \sup \{ x \geq 0 : \varphi(x) = 0 \},
\]

\[
b = \sup \{ x \geq 0 : \varphi(x) < +\infty \}.
\]

If a function \( \varphi \) is considered instead of \( \varphi \) then \( a, b \) denote the numbers \( a, b \) for the function \( \varphi \).

The connections between \( \varphi \) and \( \varphi^{-1} \) are formulated in the following simple lemma.

1. **Lemma.** For all \( x \in [0, +\infty] \)

\[
x \leq \varphi^{-1}(\varphi(x)) \text{ and } \varphi(\varphi^{-1}(x)) \leq x.
\]

Moreover,

\[
\varphi(\varphi^{-1}(x)) = x \text{ for } x \in [0, \varphi(b)]
\]

\[
\varphi(\varphi^{-1}(x)) = \varphi(b) \text{ for } x \in (\varphi(b), +\infty].
\]
We say that two Young functions \( \varphi, \psi \) are equivalent for large arguments (small arguments) \([\text{all arguments}]\) if

\[
\lim_{u \to \infty} \frac{\varphi(ku)}{\varphi(u)} < \infty \quad \lim_{u \to 0} \frac{\varphi(ku)}{\varphi(u)} < \infty \quad \lim_{u \to \infty} \frac{\psi(ku)}{\psi(u)} < \infty
\]

for some \( k > 0 \) and \( i, j = 1, 2 \).

In the sequel the expressions "large arguments", "small arguments", "all arguments" will be always denoted by "l.a.", "s.a.", and "a.a.", respectively.

In the rest of the paper, \( G \) will be a locally compact abelian group, with Haar measure \( \mu \). Let \( \mathcal{F} \) the family of all \( \mu \)-measurable, complex valued functions \( f \) defined on \( G \). The Orlicz space \( L_\varphi \) is a modular space generated by the modular

\[
\lambda(f) = \int |f(t)|^\varphi(t) \, d\mu(t)
\]

The subspace of finite elements of the space \( L_\varphi \) is denoted by \( E_\varphi \). It is well known that, when \( \varphi \) is finite, \( E_\varphi \) consists of those elements of \( L_\varphi \) which are order continuous (\([6],[5]\)). Let us recall that if \( \varphi_1 \) and \( \varphi_2 \) are equivalent for l.a. (s.a.) \([a.a.]\) then \( L_{\varphi_1} = L_{\varphi_2} \) when \( G \) is nondiscrete and compact \((G \) is nondiscrete and noncompact) \([G \) is discrete]. If \( \varphi_1 \) and \( \varphi_2 \) are finite then equivalence of these functions implies also that \( E_{\varphi_1} = E_{\varphi_2} \).

In further considerations the important role will be played by the following two conditions. Let \( \varphi_i, i = 1, 2, 3 \), be Young functions.

It is said that \( \varphi_i \) satisfy condition \((+)\) for l.a. (s.a.) \([a.a.]\) if there exist \( k > 0 \), \( \delta > 0 \) such that

\[
kuv \leq \varphi_i(u)\varphi_1^{-1}(\varphi_i(v)) + \varphi_2(v)\varphi_2^{-1}(\varphi_i(u))
\]

when \( \varphi_i(u) \geq \delta \) and \( \varphi_2(v) \geq \delta \) \((\varphi_i(u) < \delta \) and \( \varphi_2(v) < \delta \))[\(u,v \geq 0\)].

It is said that \( \varphi_i \) satisfy condition \((++)\) for l.a. (s.a.) \([a.a.]\) if for every \( \alpha > 0 \) there exist \( k > 0 \), \( \delta > 0 \) such that

\[
\alpha uv \leq \varphi_i(u)\varphi_1^{-1}(k\varphi_i(v)) + \varphi_2(v)\varphi_2^{-1}(k\varphi_i(u))
\]

when \( \varphi_i(u) \geq \delta \) and \( \varphi_2(v) \geq \delta \) \((\varphi_i(u) < \delta \) and \( \varphi_2(v) < \delta \))[\(u,v \geq 0\)].

The above conditions can be reformulated equivalently. Namely, we have the following proposition. The proof will be omitted because it is analogous to that of Lemma 2.4 in \([8]\).
2. Proposition.

- Condition (+) for l.a. (s.a.) [a.a.] is equivalent to the following one:

there exist \( l, \delta > 0 \) such that

\[
\varphi_i^{-1}(u) \varphi_j'^{-1}(u) \leq lu \varphi_i'^{-1}(u)
\]

if \( u \geq \delta (u \leq \delta) [u \geq 0] \).

- Condition (++) for l.a. (s.a.) [a.a.] is equivalent to the following one:

for every \( a > 0 \) there exist \( l, \delta > 0 \) such that

\[
\varphi_i'^{-1}(u) \varphi_j'^{-1}(u) \leq au \varphi_i'^{-1}(lu)
\]

if \( u \geq \delta (u \leq \delta) [u \geq 0] \).

- Condition (+) is invariant under equivalence of the functions \( \varphi_i \), which is shown in the next proposition.

3. Proposition. If \( \varphi_i \) satisfy condition (+) for l.a. (s.a.) [a.a.] and \( \varphi_j \), are equivalent to \( \varphi_i \) for l.a. (s.a.) [a.a.], then \( \varphi_j \) satisfy condition (+) for l.a. (s.a.) [a.a.] again.

Proof. For instance, we shall show that \( \varphi_j \) satisfy condition (+) for s.a. Since \( \varphi_j \) are equivalent to \( \varphi_i \), there exist \( \delta, l > 0 \) such that \( \varphi_i(lu) \leq \varphi_i(u) \) if \( \varphi_i(u) \leq \delta \), \( i = 1, 2 \) and \( \varphi_j(lu) \leq \varphi_j(u) \) if \( \varphi_j(u) \leq \delta \). Put \( l = \min_l \) and \( \delta = \min \delta \). Without loss of generality suppose \( \delta \leq \delta \), where \( \delta \) is the constant from condition (+). Then, by condition (+) we have

\[
kPuv \leq \varphi_i(lu) \varphi_j'^{-1}(\varphi_j^{-1}(lv)) + \varphi_j'(lv) \varphi_j'^{-1}(\varphi_j^{-1}(lu)) \leq \varphi_i(u) \varphi_j'^{-1}(\varphi_j^{-1}(u)) + \varphi_j(u) \varphi_j'^{-1}(\varphi_j^{-1}(u))
\]

if \( \varphi_j(u) \leq \delta \) and \( \varphi_j(v) \leq \delta \).

Since \( \varphi_j(lu) \leq \varphi_i(u) \) when \( \varphi_i(u) \leq \delta \), so \( \varphi_j'^{-1}(\varphi_j^{-1}(u)) \leq \varphi_i'^{-1}(\varphi_i^{-1}(u)) \) for \( v \leq \delta \). But for \( \delta > v = \varphi_i(u) > 0 \), \( u = \varphi_j'^{-1}(\varphi_j^{-1}(u)) = \varphi_j'^{-1}(v) \) and so \( \varphi_i'^{-1}(v) \leq (1/l) \varphi_j'^{-1}(v) \). If \( v = \varphi_i(u) = 0 \) then the inequality is also true because \( \varphi_j^{-1}(a_i) \leq \varphi_i^{-1}(a_i) = 0 \) implies \( lu \leq \delta \), i.e. \( \varphi_i'^{-1}(0) \leq (1/l) \varphi_j'^{-1}(0) \). Thus

\[
kPuv \leq (1/l) \varphi_i(u) \varphi_j'^{-1}(\varphi_j^{-1}(v)) + (1/l) \varphi_j(v) \varphi_j'^{-1}(\varphi_j^{-1}(u))
\]
If $\bar{\phi}_1(u) \leq \delta_u$ and $\bar{\phi}_2(v) \leq \delta_v$, which means that $\bar{\phi}$ satisfy condition (+) for s.a.

Now we shall discuss the case of a discrete group $G$ in the connection with the Orlicz space and its subspace of finite elements. Traditionally, in this case the notations $L_p$ and $h_p$ are used instead of $L_p$ and $E_p$. First let us note the following simple fact.

4. **Lemma.** For every Young function $\Phi$ there exists a Young function $\Phi$ finite and equivalent to $\Phi$ for s.a.

As a corollary, it is seen that instead of $L_p$ where $\Phi$ can take infinite values, one can always consider the isomorphic space $L_{ps}$ where $\Phi$ is finite. But there are some problems with the subspace of finite elements. If $\Phi$ is infinite for some real numbers, then $h_p = \{0\}$, whereas $h_p$ is always different from $\{0\}$ if $\Phi$ is finite. Thus an equivalent function $\bar{\Phi}$ defines a different subspace of finite elements than the function $\Phi$. However, let us note that for any function $\Phi$ there exists the only subspace of finite elements defined by a function $\Phi$ finite and equivalent to $\Phi$. This subspace $h_p$ does not depend on the choice of the function $\Phi$, belonging to the class of all Young functions finite and equivalent to $\Phi$.

Taking into consideration the above remarks, in the sequel we shall always assume that $\Phi$ is finite in the case of a discrete group.

The Lemmas 5, 6 and 8 are some technical steps to prove Theorems 7 and 9.

5. **Lemma.** If $\Phi$ are finite and satisfy condition (+) for s.a., then there exist functions $\Phi$, finite and equivalent to $\Phi$, for s.a. satisfy condition (+) for a.a., if $\Phi$ satisfy condition (+) for l.a., then there exist functions $\Phi$, equivalent to $\Phi$, for l.a. and satisfying condition (+) for a.a.

**Proof.** Let first $\Phi$ satisfy condition (+) for s.a. Put $h(u,v) = \Phi_1(u)\Phi_2(v) + \Phi_2(v)\Phi_1(u)$ and $h(u,v)$ if $\Phi$, are replaced by $\tilde{\Phi}$. Let $u_0$ be such that $\Phi(u_0) = \delta$ and put

$$
\Phi(u) = \begin{cases} 
\Phi(u), & u \in [0,u_0] \\
\Phi_1'(u_0)u + \Phi(u) - \Phi_1'(u_0)u, & u \in (u_0,\infty).
\end{cases}
$$

where $\Phi_1'$ is a right-hand derivative of $\Phi$. We have $\Phi(u) \geq \bar{\Phi}(u)$ and $\Phi_1'(v) \leq \bar{\Phi}_1'(v)$ for all $u, v \geq 0$. We shall show that $\bar{\Phi}$ satisfy condition (+) for a.a. For $u \leq u_0$, $v \leq u_0$ the inequality is immediate. Let $u \geq u_0$, and $v \geq u_0$. Then we can write $\Phi(u) = c_1u + d$, for $u \geq u_0$, where $c_1 > 0$ and $d < 0$. Hence we simply obtain
where $M$ is a constant dependent on $c, d$. Since $c, c, c > 0$, there exist $e, > 0$ and $u, > \max(u_0, u_2)$ such that
\[ \tilde{\phi}(u)\tilde{\phi}^{-1}(\tilde{\phi}(v)) \geq e, uv \] whereas $u, v \geq u$. Moreover,
\[ \frac{\tilde{\phi}(u)\tilde{\phi}^{-1}(\tilde{\phi}(v))}{uv} > \frac{\delta u_0}{u_0^2} > 0 \]
for $u \in [u_0, u_2]$ and $v \in [u_0, u_2]$. Then for $e = \min \left( e, \frac{\delta u_0}{u_0^2} \right)$ and $u \geq u_1$, $v \geq u_1$ we have $h(u, v) \geq e, uv$.

Let now $u \geq u_1$ and $v \leq u_2$. Then
\[ h(u, v) \geq (\phi_1^{-1}(\phi_2(v)) + \phi_2(v)) \min(c, u + d, - \frac{c_1}{c_3} u + \frac{d_1}{c_3}) \]
However, by the assumption $\phi_1$ we have
\[ ku_1 \leq \delta \phi_1^{-1}(\phi_2(v)) + \phi_2(v) u_1 \] for $v \leq u_2$. Hence
\[ \max(\phi_1^{-1}(\phi_2(v)), \phi_2(v)) \geq e, v \]
where
\[ e, = \min(k/2\delta)u_1, ku_1/2u_2 \] $> 0$. Therefore
\[ h(u, v) \geq e, v \min(c, u + d, - \frac{c_1}{c_3} u + \frac{d_1}{c_3}) \]
for $u \geq u_1$ and $v \leq u_2$. Since $\tilde{\phi}(u_0) = \delta > 0$ and
\[ \tilde{\phi}^{-1}(\tilde{\phi}(u_0)) = u_0 > 0 \] and the functions $\tilde{\phi}(u)$ and $\tilde{\phi}^{-1}(\tilde{\phi}(u))$ are linear for $u \geq u_1$, there exists a constant $e, > 0$ such that
\[ \min(c, u + d, - \frac{c_1}{c_3} u + \frac{d_1}{c_3}) \geq e, u \] for $u \geq u_1$.

Hence $h(u, v) \geq e, e, uv$ when $u \geq u_1$ and $v \geq u_2$. So we proved the first part of the lemma.

Now let $\phi_1$ satisfy condition $\phi_1$ for l.a. It is not difficult to verify that the functions
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\[ \tilde{\phi}(u) = \begin{cases} 
\phi(u) & \text{if } \phi(u) > \delta \\
(\delta/\phi) & \text{if } \phi(u) \leq \delta 
\end{cases} \]

satisfy condition (+) for a.a. Moreover, it is evident that they are equivalent to \( \phi \), for l.a. which finishes the proof of the lemma.

6. **Lemma.** If \( \phi \) satisfy condition (+) for a.a. and \( I_\phi(f) \leq 1 \) and \( I_\phi((2\lambda/k)f) \leq 1 \) (or \( I_\phi(f) \leq 1 \) and \( I_\phi((2\lambda/k)g) \leq 1 \) where \( k \) is the constant from (+), then \( I_\phi(fg) < \infty \).

**Proof.** Applying (+) we obtain

\[
I_\phi(\lambda f g) \leq \int \phi(\lambda f g(t)) \phi^{-1}(\phi(\lambda f g(t))) \, d\mu(t) \\
+ 1/2 \int \phi(g(t^{-1}x)) \phi^{-1}(\phi(g(t^{-1}x))) \, d\mu(t) \\
\leq 1/2 \int \phi(\lambda f g(t)) \phi^{-1}(\phi(\lambda f g(t))) \, d\mu(t) \\
+ 1/2 \int \phi(g(t^{-1}x)) \phi^{-1}(\phi(g(t^{-1}x))) \, d\mu(t)
\]

Since \( I_\phi(2\lambda/k) \leq 1 \) and \( I_\phi(g) \leq 1 \), by Jensen's inequality
\[
I_\phi(\lambda f g) \leq I_\phi(2\lambda/k) I_\phi(g) \leq 1 < \infty.
\]

The next theorem gives sufficient conditions for embeddings of the spaces \( L_{\phi_1} \hookrightarrow L_{\phi_2} \) into \( L_{\phi_1} \) and \( L_{\phi_1} \).

7. **Theorem.** 1. Let \( G \) be nondiscrete and \( \phi \) satisfy condition (+) for l.a. if \( G \) is compact and (+) for a.a. Then \( L_{\phi_1} \hookrightarrow L_{\phi_2} \). If additionally \( \phi_1 \) is finite, then \( E_{\phi_1} \hookrightarrow E_{\phi_2} \).

II. Let \( G \) be discrete and \( \phi \) satisfy condition (+) for s.a. Then \( h_{\phi_1} \hookrightarrow h_{\phi_2} \) and \( l_{\phi_1} \hookrightarrow l_{\phi_2} \).

**Proof.** By Theorem 1.2 it is enough to prove only inclusions. I. Let first \( G \) be noncompact and \( \phi \) satisfy (+) for a.a. The proof of the inclusion \( L_{\phi_1} \hookrightarrow L_{\phi_2} \) is an immediate consequence of the previous lemma. Really, taking \( f \in L_{\phi_1} \) and \( g \in L_{\phi_2} \) from the unit balls we have \( I_{\phi_1}(f) \leq 1 \) and \( I_{\phi_1}(g) \leq 1 \). So we can apply the lemma with \( \lambda = k/2 \) and thus \( I_{\phi_1}(k/2 f g) < \infty \), which means that \( f g \in L_{\phi_1} \).

To prove the inclusion \( E_{\phi_1} \hookrightarrow L_{\phi_1} \) take \( f \in E_{\phi_1} \), \( g \in E_{\phi_2} \). Let \( \lambda = 2\beta \), where \( \beta \) is arbitrary. Since the Haar measure \( \mu \) is regular and \( I_{\phi_1}(2\lambda/k f) < \infty \) and...
there exist compact sets \( G_1, G_2 \) such that \( I_{\phi}(2\lambda/k) \leq 1 \) and \( I_{\phi}(2\lambda/k \, g \alpha_0) \leq 1 \). We can write

\[
I_{\phi}(2\lambda/k \, g) \leq 1 + 1/3I_{\phi}(\lambda f \alpha_0 \, g \alpha_0) + 1 \]

By the previous lemma, the first two components of the above inequality are finite, so it is enough to show that \( I_{\phi}(\lambda f \alpha_0 \, g \alpha_0) < \infty \). Since the support of \( f \alpha_0 \, g \alpha_0 \) is contained in \( G_0 \),

\[
I_{\phi}(\lambda f \alpha_0 \, g \alpha_0) \leq \int_{\mathbb{R}^n} |f(t)| |g(t^{-1})| \phi(t) \, dt \mu(t) \mu(x). 
\]

There exists \( u_0 \geq 0 \) such that

\[
(8.2) \quad I_{\phi}(\lambda f \alpha_0 \, g \alpha_0) \leq \int_{\mathbb{R}^n} |f(t)| |g(t^{-1})| \phi(t) \, dt \mu(t) \mu(x). 
\]

Denote \( G_\phi = G_1 \cap G_2 \). We have

\[
\int_{\mathbb{R}^n} |f(t)| |g(t^{-1})| \phi(t) \, dt \mu(t) \mu(x) 
\]

since every function from Orlicz space is locally integrable. Analogously

\[
\int_{\mathbb{R}^n} |f(t)| |g(t^{-1})| \phi(t) \, dt \mu(t) \mu(x) \leq M_1 < \infty. 
\]

Thus,

\[
I_{\phi}(\lambda f \alpha_0 \, g \alpha_0) \leq \int_{\mathbb{R}^n} |f(t)| |g(t^{-1})| \phi(t) \, dt \mu(t) \mu(x) \leq 
\]

Denoting by \( M_1 \) the first component of the above sum and applying condition (+) to the second one, we get

\[
I_{\phi}(\lambda f \alpha_0 \, g \alpha_0) \leq M_1 + 
\]

\[
+ 1/2|f_{\phi}(\alpha_0, \phi_1(g(t^{-1})) \phi_2'(\phi_1(g(t^{-1}))) \mu(t) \mu(x) 
\]

Thus,

\[
I_{\phi}(\lambda f \alpha_0 \, g \alpha_0) \leq M_1 + 
\]

\[
+ 1/2|f_{\phi}(\alpha_0, \phi_1(g(t^{-1})) \phi_2'(\phi_1(g(t^{-1}))) \mu(t) \mu(x) 
\]
In virtue of (8.2) and Jensen's inequality,

\[ I_{\lambda}(\lambda f_{\phi_{\gamma}}*g_{\phi_{\gamma}}) \leq M_{\lambda} + \frac{1}{2} \int_{\Omega} \int_{\Omega} \frac{1}{k^{\gamma}(t)} \phi_{\gamma}((g(t^{-1}x)) \phi_{\gamma}((t^{-1}x)) d\mu(t) d\mu(x) + \frac{1}{2} \int_{\Omega} \int_{\Omega} \frac{1}{k^{\gamma}(t)} \phi_{\gamma}((g^{(t^{-1}x)}) \phi_{\gamma}((t^{-1}x)) d\mu(t) d\mu(x) \leq M_{\lambda} + I_{\lambda}(6\lambda/k)I_{\lambda}(g) < \infty. \]

If \( G \) is compact and \( \phi \) satisfy condition \((+)\) for l.a., then by Lemma 5 there exist functions \( \tilde{\phi} \) satisfying \((+)\) for a.a. and such that \( E_{\tilde{\phi}} = E_{\phi} \), \( L_{\tilde{\phi}} = L_{\phi} \). So without loss of generality one can assume that \( \phi \) satisfy condition \((+)\) for a.a. Thus, the inclusions follow in the same way as above (we can put \( G_{1} = G_{2}G_{3} \)).

II. For this case, applying Lemma 5 again, we can also assume condition \((+)\) for a.a. In the inequality (8.1) we can see analogously as in I that the first two components are finite. The third component is also finite since the support of \( f_{\phi^{*_{\gamma}}*g_{\phi^{*_{\gamma}}}} \) is finite. So it is the end of the proof.

8. Lemma. If \( \phi \) satisfy condition \((++)\) for s.a. then \( l_{\phi} \subset h_{\phi} \) and \( l_{\phi} \subset h_{\phi} \).

Proof. Using the equivalent form of condition \((++)\) expressed in Proposition 2 we have \( \phi^{(-1)}(u)\phi^{(-1)}(u) < u\phi^{(-1)}(ku) \) for \( u \leq \delta \). But by concavity of \( \phi^{(-1)} \) there exists \( \delta_1 > 0 \) such that \( u/\phi^{(-1)}(u) \leq l \) for \( u \leq \delta_1 \). So \( \phi^{(-1)}(u) \leq a\phi^{(-1)}(ku) \) for sufficiently small \( u \). Putting \( r = \phi^{(-1)}(u) \) we obtain \( \phi^{(-1)}(ku) \leq k\phi^{(-1)}(r) \) for small \( r \), which immediately implies that

\[ \lim_{u \to 0} \frac{\phi^{(-1)}(\lambda u)}{\phi^{(-1)}(u)} < \infty \text{ for all } \lambda > \mathcal{Q}. \]

But the last condition implies the inclusion

\[ l_{\phi} \subset h_{\phi} \text{([9])}. \]


I. Let \( G \) be nondiscrete and \( \phi \) satisfy condition \((++)\) for l.a. if \( G \) is compact and \((++)\) for a.a. if \( G \) is noncompact. If \( \phi \) is finite, then \( L_{\phi} \subset E_{\phi} \).

II. Let \( G \) be discrete and \( \phi \) satisfy condition \((++)\) for s.a. Then \( l_{\phi} \subset h_{\phi} \).
Proof. 1. Suppose $G$ is compact. Take $f \in L^\alpha$, $g \in L^\alpha$ such that $I_{I_\lambda}(f) \leq 1$ and $I_{I_\lambda}(g) \leq 1$. Let $\lambda > 0$ be arbitrary and $\delta > 0$ be from condition $(++)$ chosen for $a=2\lambda$. Put
\[
G_1 = \{ t \in G : \varphi_r(\lambda f(t)) \geq \delta \},
\]
\[
G_2 = \{ t \in G : \varphi_r(\lambda g(t)) \geq \delta \}.
\]

The convolution $f * g$ is the sum of functions $f_{x_1} * g_{x_2}$, $f_{x_0} * g_{x_2}$, and $f_{x_0} * g_{x_2}$. Applying $(++)$ for $l.a.$ and Jensen's inequality, we get an analogously as in the proof of Theorem 7, that
\[
I_{I_\lambda}(\lambda f_{x_0} * g_{x_2}) \leq \int \varphi_r(\lambda/2 \int f(t)g(t^{-1}x)\chi_{x_2^{-1}}(t)\mu(t)d\mu(x) \leq
\int \varphi_r(1/2 \int f(t)g(t^{-1}x)\chi_{x_2^{-1}}(t))\varphi_r^{-1}(k \varphi_r(\lambda g(t^{-1}x)\chi_{x_2^{-1}}(t)))d\mu(t)
\]
\[
+ 1/2 \int \varphi_r(\lambda f(t^{-1}x)\chi_{x_2^{-1}}(t))\varphi_r^{-1}(k \varphi_r(\lambda g(t^{-1}x)\chi_{x_2^{-1}}(t)))d\mu(t)d\mu(x)
\]
\[
\leq I_{I_\lambda}(f) I_{I_\lambda}(g) < \infty.
\]

So it is enough to show that e.g. $I_{I_\lambda}(\lambda f_{x_0} * g) < \infty$.

By local integrability of $g$, we have $M = \int d\mu(t) < \infty$.

Hence $I_{I_\lambda}(\lambda f_{x_0} * g) \leq \varphi_r(\chi_{x_2^{-1}}(\delta)M)\mu G < \infty$.

In the case of noncompact $G$ the proof is similar and even simpler in the sense that $G_1 = G_2 = G$.

II. If $G$ is discrete and $f, g$ are the same as above, then there exist finite sets $G_1, G_2 \subset G$ such that $I_{I_\lambda}(f_{x_0} g_{x_2}) \leq \delta$ and $I_{I_\lambda}(g_{x_0} g_{x_2}) \leq \delta$, where $\delta > 0$ is the constant from condition $(++)$ chosen for $a=2\lambda > 0$. We have
\[
I_{I_\lambda}(f_{x_0} g_{x_2}) = (f_{x_0} * g_{x_2}) + (f_{x_0} * g_{x_2}) + (f_{x_0} * g_{x_2}) + (f_{x_0} * g_{x_2}).
\]

The first component belongs to $h_0$ because its support is finite. Applying $(++)$ and Jensen's inequality to the last one similarly as in I, we get $I_{I_\lambda}(f_{x_0} g_{x_0}) \leq k < \infty$.

To finish the proof note that
\[
\chi(\lambda f_{x_0} g_{x_2})(x) \leq \sum_{i \in I_\lambda} \int f(t)g(t^{-1}x)\chi_{x_2^{-1}}(t)
\]
for every $x \in G$, where $g(t^{-1}) \in L^\alpha$, for all $t \in G$. Thus
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$$\sum_{t \in \mathbb{Q}} |f(t)||g(t^{-1})| \in L_2$$ and so $f_{\alpha,1}g_{\alpha,2} \in L_2$.

But by Lemma 8, we have $L_2 \subset h_2$, so $f_{\alpha,1}g_{\alpha,2} \in h_2$.

Next theorems will be converse to the results obtained in Theorem 7.

10. Theorem. Let $G$ be discrete. If $h_1 \ast h_2 \subset L_2$ then $G$ is finite or condition (+) for s.a. is satisfied.

Proof. Remember, in the sequence case we have assumed the functions $\varphi$ have been finite. For a contrary, suppose the group $G$ is infinite and condition (+) for s.a. is not satisfied. There exist sequences $(u_n),(v_n)$ such that $\varphi(u_n) \to 0$ and $\varphi(v_n) \to 0$ and

$$\frac{1}{n} u_n v_n > \varphi(u_n) \varphi^{-1}(\varphi(v_n)) + \varphi(v_n) \varphi^{-1}(\varphi(u_n)).$$

Without loss of generality assume $\varphi(u_n) \geq \varphi(v_n)$.

We shall consider a few cases.

I. Let $\varphi(u_n) \geq \varphi(v_n) > 0$. Let $u_n$ be such that $\varphi(u_n) = \varphi(v_n)$. Since $\varphi(u) / u$ is nondecreasing, $\frac{1}{n} u_n v_n > \varphi(u_n) \varphi^{-1}(\varphi(v_n))$, by (10.1). So we can assume about $u_n v_n$ that $\varphi(u_n) = \varphi(v_n)$ and

$$\frac{1}{n} u_n v_n > \varphi(u_n) \varphi^{-1}(\varphi(v_n)).$$

We shall examine two types of the group $G$.

(a). Let $G$ contain a cyclic subgroup of arbitrary large rank.

There exist natural numbers $l_n$ such that

$$\frac{1}{2l_n + 1} \varphi(u_n) \leq 1 \text{ and } (2l_n + 3) \varphi(u_n) > 1.$$

Take a cyclic subgroup $S$ such that $rS > 2l_n + 1$. Let

$$A_n = \{ r \in S : i = 0, 1, \ldots, l_n - 1, \ldots, -l_n \},$$

and

$$f_n(t) = u \chi_n(t), \quad g_n(t) = v \chi_n(t).$$
By (10.3), it is evident that

\[(10.4) \quad \frac{1}{2} \leq I_*(f_*) \leq 1, \quad \frac{1}{2} \leq I_*(g_*) \leq 1.\]

Moreover

\[(f_*g_*)(x) = u_*v_\mu(A_* \cap A_* x).\]

If \(x \in A_*\), then either \(A_* \cap A_* x = \{e, t, \ldots, t^k\}\) or \(A_* \cap A_* x = \{e, t^{-1}, \ldots, t^{-k}\}\). Therefore \(\mu(A_* \cap A_* x) = l_* + 1\) for \(x \in A_*\).

Hence

\[(10.5) \quad (f_*g_*)(x) \geq u_*v_\mu(l_* + 1)\chi_\mu(x).\]

But by (10.3), we get the following estimation of \(l_*\)

\[(10.6) \quad l_* + 1 \geq 1/3\phi_\mu(u_*).\]

Thus in virtue of (10.2)

\[
\frac{3}{n} (f_*g_*)(x) \geq \frac{1}{n} u_*v_\mu \frac{l_*}{\phi_\mu(u_*)} \chi_\mu(x) > \phi^{-1}_\mu(\phi_\mu(v_\mu))\chi_\mu(x).
\]

Hence and by Lemma 1 and the fact that \(\phi_\mu(h) = \infty\) and by (10.4), we have

\[I_*(3/n f_*g_*) > \phi_\mu(v_\mu)\mu A_* = I_*(g_*) \geq 1/2.\]

So we found sequences \(f_* \in h_*, g_* \in h_*\) such that \(I_*(f_*) \leq 1, \quad I_*(g_*) \leq 1\) and \(I_*(3/n f_*g_*) \geq 1/2\). Applying Theorem 1.2 we can see that \(h_* \neq h_*, g_* \neq g_*\).

(b). Let, contrary to (a), the rank of all cyclic subgroups of \(G\) be bounded. So there exists a prime number \(k\) and infinite number of cyclic subgroups with rank equal to \(k\). Let \(S_i\) be an infinite sequence of cyclic subgroups such that \(rS_i = k, S_i \cap S_j = \{e\}\) for \(i \neq j\). Let \(P_* = \bigoplus \{S_i\}\) be a simple sum of \(S_1, \ldots, S_n\). The set \(P_*\) is a subgroup of \(G\) containing \(k^n\) different elements. Put

\[l_* = \log_\phi_\mu(u_*) \quad \text{and} \quad f_*(t) = u_*\chi_{h_*}(t) \quad \text{and} \quad g_*(t) = v_{\mu} \chi_{h_*}(t).
\]

It is clear that
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(10.7) \[ 1/k \leq I_{\Phi}(f) \leq 1, \quad 1/k \leq I_{\Phi}(g) \leq 1. \]

Moreover,

\[ (f * g)(x) \geq u_{\Psi} \Phi \chi_{\Theta}(x) = u \Psi \chi_{\Theta}(x). \]

But by the definition of \( l_{\omega} \) we have

(10.8) \[ k \Psi \geq 1/k \Phi(u). \]

Thus in virtue of (10.2), Lemma 1 and (10.7) we obtain

\[ I_{\Phi}(k/n f * g) \geq I_{\Phi}(g) \geq 1/k. \]

So applying Theorem 1.2 we get a contradiction.

II. Suppose \( \Phi(u) > \Phi(v) = 0 \). Hence \( a_2 > 0 \). We shall consider two cases. Let first \( a_3 = 0 \). There exists an infinite, countable subgroup \( G \) of \( G \). There exists an element \( (c_3) \in c_3 \) such that \( (c_3) \not\in l_{\Phi}(G) \). Put

\[ u(t) = \begin{cases} c_3 & \text{if } t = t_n, \quad n = 0, 1, 2, \ldots \\ 0 & \text{if } t \not\in G \end{cases} \]

\[ v(t) = \chi_{c_3}(t), \]

where \( G = \{ e, t_1, t_2, \ldots \} \). Since \( a_2 > 0 \), \( h_{c_3} \) is isomorphic to \( c_3 \). So it is clear that \( u \in h_{c_3} \), \( v \in h_{c_3} \) and \( u * v = 0 \).

Now let \( a_3 > 0 \). In this case we modify the proof of the part I. Let \( C_{\Phi} \) denote \( A_3 \) or \( P_1 \) and \( c_3 \) denote \( I \) or \( k \) and \( c \) be equal to 3 or \( k \), respectively, when \( G \) satisfies \( (a) \) or \( (b) \). Let \( f_a(t) = u \chi_{c_3}(t) \) and \( g_a(t) = v \chi_{c_3}(t) \). Then min \( (1/2, 1/k) \leq I_{\Phi}(f) \leq I_{\Phi}(g) \leq 1 \) and

\[ (f_a * g_a)(x) \geq u \Psi \chi_{c_3}(x), \]

by (10.4), (10.7), (10.5), (10.6) and (10.8). Hence and by (10.1) we get

\[ I_{\Phi}(2 c/n f_a * g_a) \geq \int_0 \varphi(2n)(\varphi(\varphi(v))) \chi_{c_3}(x) d\mu(x) \]

\[ = \int_0 \varphi(2n) \chi_{c_3}(x) d\mu(x) = \varphi(2n) \mu C_{\Phi} \to \infty, \text{ as } n \to \infty. \]

So by virtue of Theorem 1.2 and the fact that \( f_a \in h_{c_3} \) and \( g_a \in h_{c_3} \) we get the hypothesis.
III. If \( \varphi_i(u_i) = \varphi_i(v_i) = 0 \) then \( a_i > 0 \) for \( i = 1, 2 \). So \( h_i(G_i) \) is isomorphic to \( c \), for \( i = 1, 2 \) and any infinite, countable subgroup \( G_i \) of \( G \). But it is possible to construct elements \( u, v \in c \) such that \( (u^*v)(e) = \infty \).

11. **Theorem.** For any noncompact abelian group \( G \) condition (+) for s.a. is necessary for the inclusion \( L_q \ast L_q \subset L_q \). If additionally \( \varphi_i (i = 1, 2) \) are finite then condition (+) for s.a. is necessary for the inclusion \( E_q \ast E_q \subset L_q \), too.

**Proof.** By the well known results about the structure of an abelian group either the group \( G \) contains a compact open subgroup \( G_c \) or it contains an element \( z \) such that the set \( \{ z^n : n \in \mathbb{Z} \} \) is an infinite, discrete subgroup of \( G \), where \( \mathbb{Z} \) is the set of all integers. Let \( \phi_i \) be finite and equivalent to \( \varphi_i \) for s.a. (see Lemma 4).

In the first case \( G/G_c \) is infinite and discrete. Suppose \( \mu G_c = 1 \). We shall show that \( I_q(G/G_c) \ast I_q(G/G_c) \subset I_q(G/G_c) \). For \( f \in I_q(G/G_c) \) and \( g \in I_q(G/G_c) \) put \( \tilde{f}(x) = f(xG_c) \) and \( \tilde{g}(x) = g(xG_c) \) where \( xG_c \) belongs to \( G/G_c \). Then \( f \in I_q(G/G_c) \), \( g \in I_q(G/G_c) \) and clearly \( f \ast \tilde{g} \in I_q(G/G_c) \). By the assumption \( f \ast \tilde{g} \in I_q(G/G_c) \). But \( f \ast \tilde{g} = f \ast g(xG_c) \) for all \( x \in G_c \), because \( \mu G_c = 1 \). Therefore \( f \ast \tilde{g} \in I_q(G/G_c) \). Thus by virtue of finiteness of \( \phi_i \) and Theorem 10, \( \phi_i \) satisfy condition (+) for s.a. But Proposition 3 implies that \( \varphi_i \) satisfy condition (+) for s.a., too.

In the second case denote \( G_c = \{ z^n : n \in \mathbb{Z} \} \). Analogously as above it is enough to show that \( I_q(G_c) \ast I_q(G_c) \subset I_q(G_c) \). Take arbitrary \( a = (a_i) \in I_q(G_c) = I_q(G_c) \) and \( b = (b_i) \in I_q(G_c) = I_q(G_c) \). Then \( a \ast b = c \) where \( c_i = \sum a_i b_{-i} \).

Let \( U, V \) be symmetric neighbourhoods of \( e \) such that \( U \cap G_c = \{ e \} \) and \( V \subset U \). Put \( f(t) = \sum_{i \neq z} a_i x_i e(t), g(t) = \sum_{i \neq z} b_i x_i e(t) \). Clearly \( f \in L_{q_1} \), \( g \in L_{q_2} \) and so \( f \ast g \in L_{q_1} \). Moreover,

\[
|f|^{q_1} |g(x)| \geq \sum_{i \neq z} |a_i| \mu_j x_i \mu(Uxz^{-n} \cap U).
\]

If \( x \in V \) then \( xz^{-n} \cap U \). Hence \( \mu(Uxz^{-n} \cap U) \geq \mu V \). Thus \( f \ast g(x) \chi_{x \cap c} + c_i \mu V \). Therefore there exists \( \lambda > 0 \) such that \( \sum q_i (\lambda \mu V c_i) \leq I_q(\lambda \mu V c_i) \mu(g) < \infty \), which shows that \( c = (c_i) \in I_q(G_c) = I_q(G_c) \).

Thus the first part of the theorem is proved. The proof of the second one is similar and even simpler because the functions \( \varphi_i \) are finite by the assumption.
Now, let us introduce two conditions for a locally compact group.

We say that a group $G$ satisfies condition (*) if for every sequence $a_i \to \infty$ there exist sequences $(U_i), (V_i)$ of measurable sets and constants $\kappa, k_1, k_2 > 0$ such that

1. $k_i \leq a_i \mu U_i \leq k_{i+1}$
2. $V_i V_i^{-1} \subset U_i$
3. $\mu U_i \leq \kappa \mu V_i$

for every $i \in \mathbb{N}$.

It is said that a sequence $(U_i, V_i)$ is a so called $D^\kappa$-sequence ([2]) if $U_i, V_i$ are measurable sets and there exists $\kappa > 0$ such that

1. $U_i \supseteq U_j \supseteq \ldots, U_i \to e$
2. $V_i V_i^{-1} \subset U_i$
3. $\mu U_i \leq \kappa \mu V_i$

for every $i \in \mathbb{N}$.

Note, in the above two conditions we may always suppose that $V_i \subset U_i$.

12. Remark. The following groups satisfy condition (*): $(\mathbb{R}, +), (\mathbb{K}, +), (T, \cdot), (\mathbb{R} \setminus \{0\}, \cdot), (\mathbb{K} \setminus \{0\}, \cdot)$, where $T$ is a subgroup of $(\mathbb{K} \setminus \{0\}, \cdot)$ consisting of all elements belonging to the unit sphere of $\mathbb{K}$.

In [2] there are examples of groups admitting a $D^\kappa$-sequence. For instance the groups containing an open subgroup of the form $\mathbb{R}^m \times T \times F$, where $a, b$ are positive integers and $F$ is a finite group, admit a $D^\kappa$-sequence.

13. Proposition. If a group $G$ contains an infinite, discrete and cyclic subgroup and $G$ admits a $D^\kappa$-sequence, then the condition (*) is satisfied.

Proof. Let $(\alpha_i), \alpha_i \geq 1$, be an arbitrary sequence tending to infinity and $\{z^n : n \in \mathbb{Z}\}$ be a discrete subgroup of $G$. If $W$ is a neighbourhood of $e$ such that $\{z^n W\}$ is a pairwise disjoint family of sets, then we may assume that $U_i \subset W$. 

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where \((U, V)\) is a \(D^*\)-sequence. We find a subsequence \((U_j)\) and natural numbers \(k_j\) such that \(1/2 \leq \alpha_j(2k_j + l)\mu U_j \leq l\). Putting

\[
P_j = \bigcup_{n=2k_j} \bigcup_{l=2k_j} z^nU_j, \quad Q_j = \bigcup_{n=2k_j} \bigcup_{l=2k_j} z^nV_j,
\]

we have

\[
\mu P_j = (4k_j + l)\mu U_j \quad \text{and} \quad Q_j = (2k_j + l)\mu V_j.
\]

Hence

\[
1/4 \leq \alpha_j \mu P_j \leq 2 \quad \text{and} \quad \mu P_j \leq 2k_j \mu Q_j.
\]

Moreover,

\[
Q_jQ_j = \bigcup_{n=2k_j} \bigcup_{l=2k_j} z^nV_j = \bigcup_{l=2k_j} z^nV_j = P_j,
\]

Thus the group \(G\) satisfies condition \((\ast)\).

14. Theorem. Let a group \(G\) satisfy condition \((\ast)\). Then condition \((\ast)\) for l.a. is necessary for the inclusion \(L_{k_1} \ast L_{k_2} \subset L_{k_3}\).

If moreover \(\varphi(i=1, 2)\) are finite, then the condition \((\ast)\) for l.a. is necessary for the inclusion \(E_{k_1} \ast E_{k_2} \subset L_{k_3}\), too.

Proof. Assume \(\varphi(i=1, 2)\) are finite (in another case condition \((\ast)\) for l.a. is always satisfied). Suppose condition \((\ast)\) for l.a. is not satisfied. Then there exist sequences \((u_i), (v_i)\) such that \(\varphi_i(u_i) \to \infty\) and \(\varphi_i(v_i) \to \infty\) and

\[
I/\int u_i v_i > \varphi_i(u_i)\varphi_i^{-1}(\varphi_i(v_i)) + \varphi_i(v_i)\varphi_i^{-1}(\varphi_i(u_i)).
\]

Analogously as in the proof of Theorem 10 one can put \(\varphi_i(u_i) = \varphi_i(v_i)\). By the assumed condition \((\ast)\), one can find a sequence \(U_i\) of measurable sets such that

\[
k_1 \leq \varphi_i(u_i)\mu U_i \leq k_2,
\]

where \(k_1, k_2 > 0\). Putting

\[
f_i(t) = u_i \chi_{U_i}(t), \quad g_i(t) = v_i \chi_{U_i}(t),
\]

we have

\[
f_i^*g_i(x) = u_i \nu(xU_i \cap U_i).
\]

We can assume that \(V_i \subset U_i\), where \(V_i\) are sets from condition \((\ast)\). So, if \(x \in V_i\), then \(xV_i \subset U_i\) and
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\[
\mu(xU_i \cap U_j) \geq \mu(xU_i \cap xV_i) = \mu_{V_i}
\]
for \( x \in V_i^{-1} \). Thus

\[
f_i \ast g(x) \geq u_i \cdot \mu V_{xU_i^{-1}}(x).
\]

Then in virtue of (14.1) and (14.2), we get

\[
\frac{\mu}{\mu_k} \frac{1}{i} (f_i \ast g_i)(x) \geq \frac{\mu}{\mu_k} \phi_i \varphi_i^{-1}(\varphi_i(x)) \mu V_{xU_i^{-1}}(x) \geq \varphi_i^{-1}(\varphi_i(x)) \mu U_i.
\]

Therefore

\[
I_{\epsilon_k} \left( \frac{\mu}{\mu_k} \frac{1}{i} f_i \ast g_i \right) \geq \frac{1}{\kappa} \phi_i \varphi_i^{-1}(\varphi_i(x)) \mu U_i.
\]

Now if \( b_i = \infty \) then

\[
I_{\epsilon_k} \left( \frac{\mu}{\mu_k} \frac{1}{i} f_i \ast g_i \right) \geq \frac{k_i}{\kappa} \quad \text{for every } i \in \mathbb{N},
\]

if \( b_i < \infty \) then \( 4\varphi_i^{-1}(\varphi_i(x)) \geq 2b_i \) for sufficiently large \( i \) and so

\[
I_{\epsilon_k} \left( \frac{4\kappa}{\kappa_i} \frac{1}{i} f_i \ast g_i \right) \geq \varphi_i(2b_i) \mu U_i = \infty.
\]

Thus we have found sequences \((f_i, g_i)\) such that \( f_i \in E_{\epsilon_k}, g_i \in E_{\epsilon_k} \) and \( I_{\epsilon_k}(f_i) \leq k_i \), \( I_{\epsilon_k}(g_i) \leq k_i \) and \( I_{\epsilon_k}(\lambda f_i \ast g_i) \geq \text{const} \) for some \( \lambda_i \to 0 \). Then, by Theorem 1. The inclusions \( E_{\epsilon_k} \ast E_{\epsilon_k} \subset L_{\epsilon_k} \) and \( L_{\epsilon_k} \ast L_{\epsilon_k} \subset L_{\epsilon_k} \) are not fulfilled, which ends the proof of the theorem.

The following three corollaries are immediate consequence of Theorems 1.1, 1.2, 2, 7, 10, 11 and 14.

15. Corollary. Let \( G \) be a discrete group. The following conditions are equivalent

1. \( (\varphi \ast \psi) \rightarrow (\varphi, \psi) \)
2. \( (\varphi \ast \psi) \rightarrow (\varphi, \psi) \)
3. \( (\varphi \ast \psi) \rightarrow (\varphi, \psi) \)
4. \( (\varphi \ast \psi) \rightarrow (\varphi, \psi) \)
5. \( \varphi \) satisfies condition (+) for s.a. or \( G \) is finite.
(6) There exist $l, \delta > 0$ such that
\[ \varphi_1^{-1}(u) \varphi_2^{-1}(u) \leq l \, u \, \varphi_1^{-1}(u) \]
if $u \leq \delta$, or $G$ is finite.

(7) $\|fg\|_{L_1} \leq c \|f\|_{L_1} \|g\|_{L_1}$
for some $c > 0$ and all $f \in L_1$, $g \in L_1$.

16. Corollary. Let $G$ be nondiscrete group and $\varphi_i (i = 1, 2, 3)$ be finite. Consider the conditions (1) to (4) and (7) as in Corollary 15, where $L_1$, $H_1$ are replaced by $L_\alpha$, $E_\alpha$ respectively. Moreover, let

$(5')$ $\varphi$ satisfies condition (+) for $\mu$. if $G$ is compact or for a.a. if $G$ is noncompact.

$(6')$ there exist $l, \delta > 0$ such that
\[ \varphi_1^{-1}(u) \varphi_2^{-1}(u) \leq l u \varphi_1^{-1}(u) \]
if $u \geq \delta$ and $G$ is compact or if $u \geq 0$ and $G$ is noncompact.

We have relations: (1) $\Rightarrow$ (7) and

Moreover, if a group $G$ satisfies condition (*) then $(4) \Rightarrow (5')$, i.e. all the above conditions are equivalent.

Sufficiency of the next corollary is known as Young theorem (see e.g. [2], [8]).

17. Corollary. Let $1 \leq p, q, r < \infty$.

I. If $G$ is discrete and infinite, then

\[ \frac{1}{p} + \frac{1}{q} \leq \frac{1}{r} \]
if $1/p + 1/q \geq 1/r + 1$. 

II. Let $G$ be nondiscrete compact and $1/p + 1/q \leq 1/r + 1$ or respectively $G$ is noncompact and $1/p + 1/q = 1/r + 1$, then

$$L^{*} L \hookrightarrow L.$$ 

If additionally $G$ satisfies condition (*), then the converse of the above is also true.

18. Corollary.

I (th.2 in [3]) $L_{\varphi}$ is a Banach algebra under convolution as multiplication iff $L_{\varphi} \hookrightarrow L$, i.e. $\lim_{u \to 0} \varphi(u)/u > 0$ or $G$ is compact.

II ([10]) $L^{1}(1 \leq p < \infty)$ is a Banach algebra iff $p=1$ or $G$ is compact.

Proof. I. If we put $\varphi_{i}=\varphi(i=1,2,3)$, then $\varphi_{i}$ satisfy condition (+) for l.a., by convexity of $\varphi$. Moreover, if $L_{\varphi}$ is a Banach algebra and $G$ is noncompact, then applying Theorem 11 we get condition (+) for s.a. Thus $\varphi_{i}$ satisfy (+) for a.a., which means that $\lim_{u \to 0} \varphi(u)/u > 0$. The converse is immediate, by Theorem 7.

The point I of the above Corollary (see also [3]) is the answer to the Gramsch's problem from [1], in the case of convex function $\Phi$.

References