On minimality and $l^p$-complemented subspaces of Orlicz function spaces

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ABSTRACT. Several properties of the class of minimal Orlicz function spaces $L^p$ are described. In particular, an explicitly defined class of non-trivial minimal functions is showed, which provides concrete examples of Orlicz spaces without complemented copies of $l^p$-spaces.

A classical topic in Banach spaces is the study of the existence of $l^p$-complemented subspaces. It is well-known that from the existence of $l^p$-subspaces in a Banach space $E$ does not follow that $E$ contains a complemented copy of some $l^q$-space ($1 < p < \infty$). This happens even when we restrict ourselves to reflexive Banach lattices $E$. The natural counter-examples for this are inside the class of minimal Orlicz sequence spaces studied by Lindenstrauss and Tzafriri ([L-T], [L-T2], [L-T3] pp. 164):

Theorem 1. Given $1 < \alpha \leq \beta < \infty$ arbitrary. There exists a minimal Orlicz sequence space $l^p$ with indices $\alpha$ and $\beta$ which does not have any complemented subspace isomorphic to $l^p$ for $p \geq 1$, in spite of the fact that $l^p$ contains isomorphic copies of $l^q$ for any $\alpha \leq p \leq \beta$.

Recall that an Orlicz function $F$ is minimal at 0 ([L-T]) if for every function $G \in E_{\alpha_1}$ it happens that $E_{\alpha_1} = E_{\alpha}$, where $E_{\alpha}$ is the compact set $E_{\alpha} = \{F(\lambda x) / F(\lambda) : 0 < \lambda \leq 1\}$ in $C[0,1]$. The existence of minimal functions at 0 (different of the multiplicative ones $\rho^q$) is proved by means of Zorn Lemma.

The examples given in ([L-T], [L-T]) of minimal functions are not explicitly defined in terms of elementary functions. In fact, all minimal functions are obtained, up to equivalence, via the method of constructing Orlicz functions $F_\rho$ associated to 0-valued sequences $\rho$ of $l^p$ sequences. This method due also

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EL. Hernández-B. Rodríguez-Salinas, Te Lindenstrauss and Tzafriri ([L-T], [L-T], pp. 161), is a useful technique but rather sophisticated and uneasy to handle.

One of the goals of this lecture, which collects several results in [H-R.S.], and [H-R.S.], is to present a suitable class of minimal Orlicz spaces for which the minimal functions are explicitly defined. As far as we know these functions are the first examples of non-trivial minimal functions defined in an elementary form and without appealing to the above mentioned 0-1 valued sequence method.

We refer to ([L-T], [L-T]) for the definitions and terminology used on Orlicz and Banach spaces.

The class of minimal Orlicz function spaces $L'(\mu)$ was introduced by V. Peirats and the first named author in [H-P], showing the existence of reflexive function spaces $L'(\mu)$ without any complemented copy of $p$ for any $p \neq 2$. (The Rademacher functions span a complemented subspace isomorphic to $l^p$).

Recall that a function $F$ is minimal at $\infty$ ([H-P]) if $E_f = E_{\infty}$, for every function $G \in E_f$, where $E_f$ is the compact subset of the continuous function space $C[0,\infty)$ defined by

$$E_f = \{ \frac{F(\lambda)}{F(\lambda)} : \lambda \geq 1 \}$$

This notion of minimal at $\infty$ is slightly stronger than the minimal at 0. Fixed a minimal function $M$ at 0 it is always possible to find a minimal function $F$ at $\infty$ in such a way that its restriction to the $[0,1]$ interval coincides with the function $M$.

Minimal function spaces $L'(\mu)$ have several interesting properties (see [H-P], [H-P], [P]). For instance, a minimal space $L'(0,1)$ contains always a complemented copy of the sequence space $l^p$, and moreover the projection from $L'(0,1)$ on $l^p$ is contractive. Also it holds that the associated indices to $F$ at 0 and at $\infty$ are the same, i.e. $\alpha_f = \alpha_p$ and $\beta_f = \beta_p$.

The following result was proved in [H-P] for the cases of indices placed on the same side of 2. Afterwards in [H-R.S.] this restriction was removed:

**Theorem 2.** Given $1 < \alpha \leq \beta < \infty$ arbitrary. There exists a minimal Orlicz function space $L'(0,1)$ with indices $\alpha_f = \alpha$ and $\beta_f = \beta$ which does not have any complemented subspace isomorphic to $l^p$ for any $p \neq 2$.

The proof of this result makes basically use of the fact that a minimal Orlicz function space $L'(0,1)$ contains a complemented copy of $l^p$ for $p \neq 2$ if and only if the minimal Orlicz sequence space $l^{p_f}$ does the same.
We shall show here that inside the suitable class of explicit minimal functions there are concrete examples of Orlicz (function and sequence) spaces without complemented copies of $l^p$-spaces.

Before going further, we would like to offer the motivation for the appearance of this class of functions and some related questions:

W. Johnson, B. Maurey, G. Schechtman and L. Tzafriri in ([J-M-S-T] pp. 235) consider the function $F(t) = t^p \exp(p \log t)$ for $p > 1$ where $f$ is defined by

$$f(t) = \sum_{k=1}^{\infty} (1 - \cos \frac{\pi t}{2^k}),$$

obtaining that the associated Orlicz function spaces $L^p(0,1)$ and $L^p(0,\infty)$ are isomorphic spaces. This gave a counterexample to a Mityagin's conjecture ([M]) saying that any Orlicz space (and more generally any symmetric space) with the above property has to be necessarily an $L^p$-space, $(1 \leq p \leq \infty)$. Before that, Nielsen in [N] had proved that the Mityagin conjecture is true for the restricted class of Orlicz functions with slowly variation at $\infty$.

In ([N] pp. 256) it appears also the question whether the fact that two Orlicz function spaces $L^q(0,\infty)$ and $L^q(0,\infty)$ are isomorphic implies that the corresponding Orlicz sequence spaces $l^p$ and $l^p$ have to be also isomorphic (or even more, the same space). A counterexample to this is obtained by considering the above Johnson et al. function $F$ and as $G$ the function defined by

$$G(t) = \begin{cases} 
    t^p & \text{if } 0 \leq t \leq 1 \\
    2F(t) - 1 & \text{if } t > 1 
\end{cases}$$

Then, using ([J-M-S-T], pp. 216), we have that

$$L^p(0,\infty) \approx L^p(0,1) \approx L^q(0,\infty),$$

but $l^p$ and $l^p$ are clearly not isomorphic.

When we restrict to minimal functions the above question has a positive answer:

**Proposition 3.** If $L^p(0,\infty)$ and $L^q(0,\infty)$ are isomorphic for $F$ and $G$ minimal functions then $l^p$ and $l^p$ are also isomorphic.

We present now the class of explicit minimal spaces. (In particular we get that the Johnson et al. function is minimal):

**Theorem 4.** Given $p > 1$ and $q$ arbitrary. If $F_{\mu}$ is the function $F_{\mu}(0) = 0$ and
then $L^{\infty}(\mu)$ is a minimal Orlicz space.

Sketch of the Proof: First notice that for $q = 0$ we get the $L^r$-spaces, so the result is obvious.

Let us consider $F_q \equiv F$ for $q \neq 0$. If $G \in E_{F_q}$, and $G$ is not equivalent to $F$, there exists a sequence $(s_n)_{n=1}^\infty$, such that

$$G(t) = \lim_{n \to \infty} \frac{F(e^{s_n} t)}{F(e^t)} = e^{e^{s_n} t}$$

uniformly on the compact subsets of $[0, \infty)$ and where the function $g$ is defined by

$$g(x) = \lim_{n \to \infty} \left[ f(s_n + x) - f(s_n) \right]$$

and

$$= \lim_{n \to \infty} \sum_{k=1}^n \left( \cos \frac{\pi s_n}{2^k} - \cos \frac{\pi (x + s_n)}{2^k} \right).$$

Now for each $m \in \mathbb{N}$ we can take an scalar $0 \leq s^{(m)}_n \leq 2^{n+1}$ with $s_n \equiv s^{(m)}_n$ (mod. $2^{n+1}$). So, there exists a subsequence converging to a $\sigma \in [0, 2^{n+1})$. Thus, using the Cantor Diagonal method, we obtain a subsequence, denoted also by $(s_n)$, such that $s^{(m)}_n \to \sigma_n$ and $0 \leq \sigma_n \leq 2^{n+1}$ for each $m \in \mathbb{N}$.

Using the uniform convergence it can be deduced the following expression for the function $g$:

$$g(x) = \sum_{k=1}^n \left( \cos \frac{\pi \sigma_n}{2^k} - \cos \frac{\pi (x + \sigma_n)}{2^k} \right).$$

Now it rests to show that the function $F \in E_{F_q}$. By considering the sequence $(r_n) = (2^{n+1} - \sigma_n)$ and the uniform convergence, it is found out that

$$\lim_{n \to \infty} g(r_n + x) - g(r_n) = \lim_{n \to \infty} \sum_{k=1}^n \left( 1 - \cos \frac{\pi x}{2^k} \right) = f(x)$$

So

$$\lim_{n \to \infty} \frac{G(e^{s_n} t)}{G(e^t)} = e^{e^{s_n} t} = F(t)$$
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and $F \in E_n$. This implies that $E_n \subseteq E_{n+1} \subseteq E_n$, and $F$ is minimal at $\infty$.

q.e.d.

A direct consequence is that the sequence spaces $\ell^\infty$ are also minimal spaces (As far as we know the first examples defined explicitly).

More properties of this class of minimal spaces are the following:

**Proposition 5.** Fixed $p > 1$. For any $q$ it holds that:

(a) The associated indices at 0 and at $\infty$ to the function $F_n$ are equal to $p$.
(b) The spaces $L^{\infty}(0,1)$ and $L^\infty(0,\infty)$ are Riesz-isomorphic.
(c) Two spaces $L^\infty$ and $L^{\ell^p}$ are isomorphic if and only if $q=r$.

The proof of (b) is analogous to ([J-M-S-T], pp. 236): The function $F_n \equiv F$ is such that there exists a constant $K > 0$ and an increasing sequence $(r_n)$ with

$$
\sum \frac{1}{F(r_n)} = 1 \quad \text{and} \quad K^{-1}F(t) \leq \frac{F(r_n)}{F(r_n)} \leq KF(t)
$$

for every $n \in \mathbb{N}$ and $0 \leq t < \infty$. Now, let us consider a disjoint interval sequence $(A_n)$ in $(0,1)$ with measure $\mu(A_n) = \frac{1}{F(r_n)}$ and $\phi_n$ the increasing affine mapping from $A_n$ onto $[n,n+1)$. Then the operator $T: L^\infty(0,\infty) \rightarrow L^\infty(0,1)$ defined by

$$
T(f) = \sum_{n \geq 1} r_n \hat{\phi}_n(f)
$$

is a Riesz-isomorphism.

The statement (c) is obtained using the uniqueness of the symmetric structure for reflexive Orlicz function spaces ([J-M-S-T]) and the fact that the function $f(x)$ is not bounded at $\pm \infty$.

We pass now to study the embedding of $\ell^p$ as a complemented subspace into the spaces $L^{\ell^p}$. It is still unknown a characterization of when an Orlicz (sequence or function) space contains a complemented copy of $\ell^p$. However, there exist some necessary or sufficient conditions (see [L-T], [K], [L], [H-P]).

The following definition is an extension to the function space case of the Lindenstrauss and Tzafriri's one given for the Orlicz sequence space setting:

Fixed $\sigma > 0$, the function $\nu$ is called $\sigma$-strongly non-equivalent to $E_n$ if there exist two sequences of numbers $(K_n)$ and integers $(m_n)$, so that for $n \rightarrow \infty$ $K_n \rightarrow \infty$ and $m_n = o(K_n^2)$; and $m_n$-points $t_i \in (0,1)$ such that for every $\lambda \in [\max t_i^{-1}, \infty)$ there is at least one index $i$, $1 \leq i \leq m_n$ for which
Theorem 6. Given a reflexive space $L^r(0,1)$ and $p \neq 2$. If $p'$ is $\sigma$-strongly non-equivalent to $E_r$, for some $\sigma < \frac{1}{\beta}$, then $L^r(0,1)$ does not contain a complemented copy of $l'$.

The proof of this result has two different parts. The first step is to show using the techniques developed in ([L-T,], pp. 360) that under the hypothesis of the Theorem, no weighted Orlicz sequence space $l'(w)$, with $\sum w < \infty$ (cf. [H-P,]), contains a complemented subspace isomorphic to $l'$.

The other fact needed is the following Lemma proved in [H-R.S] by using the disjointification Kadec-Pelczynski method (cf. [L,] Proposition I.c.8).

Proposition 7. Let $L^r(0,1)$ be a reflexive space. Then $L^r(0,1)$ contains a complemented copy of $l'$ for $p \neq 2$ if and only if $l'$ is isomorphic to a complemented subspace of a weighted Orlicz sequence space $l'(w)$ with $\sum w < \infty$.

Let us apply these results to the above class of minimal spaces. In order to do it we need to consider an oscillation constant $\gamma$, associated to the function

$$f(x) = \sum_{n=1}^{\infty} \left(1 - \cos \frac{\pi x}{2n}\right),$$

defined as follows

$$\gamma = \lim_{n \to \infty} \frac{\gamma_n}{n},$$

where

$$\gamma_n = \inf_{s>0} \omega_\gamma(s)$$

and

$$\omega_\gamma(s) = \max_{0 \leq x, y \leq 2^s} \left|f(x+s) - f(y+s)\right|.$$

It can be proved that $\gamma$ satisfies $0 < \gamma < 2$. The following result holds ([H-R.S]):

Theorem 8. Let $1 < p \neq 2$ and $q$ verifying that

$$\frac{p}{1+q} < \frac{\gamma}{2 \log 2}.$$
Then the space $L^{p^*}$ does not contain any complemented copy of $l^p$.

As a consequence we easily obtain a result of Lindenstrauss and Tzafriri ([L-T], pp. 163) proved by using the method of 0-1 valued sequences:

**Corollary 9.** For any $p > 1$ there exists a minimal reflexive Orlicz sequence space $L'$ with indices $a_r = b_r = p$ which does not have any complemented copy of $l^p$.

**Proof.** Fixed $p > 1$, we take $q$ as

$$q = \frac{4p \log 2}{\gamma_f}$$

Then considering the function $F_{q^*} = F$ we deduce, from Theorem 8, that $L'$ does not contain a complemented copy of $l^p$. Since $L'$ is a minimal space, we conclude that $l^p$ does not contain a complemented copy of $l^p$, either.

A natural open question is to determine values $p \neq 2$ and $q$ verifying that the Orlicz space $L^{p^*}$ contains a complemented subspace isomorphic to $l^p$.

Any positive result in this direction would imply automatically that Problem 4.b.8 in ([L-T]) has a negative solution, i.e. the existence of minimal Orlicz sequence spaces which are not prime.

Finally another open question is whether for any minimal function $F$ the associated Orlicz spaces $L'(0,1)$ and $L'(0,\infty)$ are isomorphic.

**References**


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