Primariness of some spaces of continuous functions

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Abstract. J. Roberts and the author have recently shown that, under the Continuum Hypothesis, the Banach space $l_\omega/c_\omega$ is primary. Since this space is isometrically isomorphic to the space $C(\omega^*)$ of continuous scalar functions on $\omega^*=\beta\omega-\omega$, it is quite natural to consider the question of primariness also for the spaces of continuous vector functions on $\omega^*$. The present paper contains some partial results in that direction. In particular, from our results it follows that $C(\omega^*,C(K))$ is primary for any infinite metrizable compact space $K$ (without assuming the CH).

A Banach space $X$ is said to be primary if, whenever we have a (topological) direct sum decomposition $X=E\oplus F$, then either $E$ or $F$ is isomorphic to $X$. Many Banach spaces are known to be primary; among them are the spaces $C(K)$ of continuous scalar functions on infinite metrizable compact spaces $K$ ([3],[1]). In a recent paper [2], answering a question posed by Leonard and Whitfield. James Roberts and the author have shown that, under the Continuum Hypothesis (CH), also the Banach space $l_\omega/c_\omega$ which is isometrically isomorphic to $C(\omega^*)$, is primary. (Throughtout this paper, $\omega^*$ denotes the remainder $\beta\omega-\omega$ of the Stone-Čech compactification of $\omega=\{1,2,\ldots\}$). The present paper originated from an attempt, not very successful so far, to generalize this result to the spaces $C(\omega^*,X)$, where $X$ is a Banach space.

For the purpose of this paper let us agree to say that a Banach space $X$ is nice if for every (continuous linear) operator $T:X\to X$ there exists a subspace $Y$ of $X$ which is isomorphic to $X$ and which is mapped isomorphically by one of the operators $T$ or $id_{X}-T$ onto a complemented subspace of $X$. Clearly, if $X$ is nice and $X=E\oplus F$, then either $E$ or $F$ contains a complemented isomorph of $X$.

The approach in [2] is essentially standard and consists in showing that

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(i) the space \( C = C(\omega^*) \) is nice;

(ii) (under the CH) the \( l_\omega \)-sum of (infinitely many isometric copies of) \( C \), \( l_\omega(C) = (C \oplus C \oplus \ldots)_{\omega} \), is isomorphic to \( C \);

and then proving that \( C \) is primary by an application of Pelczynski’s decomposition method.

In the present paper we first give an alternative proof of (i), and then obtain a vector analogue of (i): if \( X \) is separable and nice, then also \( C(\omega^*,X) \) is nice. We also have a vector analogue of (ii), but with a suitable modification of the \( l_\omega \)-sums used in (ii). Unfortunately, one of the crucial properties of the \( l_\omega \)-sums that makes the Pelczynski method work in [2], viz., \( l_\omega(E \oplus F) = l_\omega(E) \oplus l_\omega(F) \), does not seem to hold for our modification. In consequence, we were unable to show that if \( X \) is nice (or primary?), then \( C(\omega^*,X) \) is primary, a result which is (more or less) what one tends to expect. Nevertheless, there is something positive we can prove: If \( X \) is a separable nice Banach space which is isomorphic to its \( c_0 \)-sum, \( c_0(X) \), then \( C(\omega^*,X) \) is primary (without assuming the CH). In particular, for every infinite metrizable compact \( K \), the space \( C(\omega^*,C(K)) = C(\omega^* 	imes K) \) is primary.

Let us introduce some notation and recall some facts about \( \omega^* \). (References can be found in [2].) We denote by \( \mathcal{A} \) the algebra of clopen subsets of \( \omega^* \); \( \mathcal{A} = \{ A \in \omega^* \mid A \) is an open \( F_\sigma \)-subset of \( \omega^* \). If \( A \in \mathcal{A} \), then \( \chi_A \) denotes the characteristic function of \( A \) relative to \( \omega^* \), \( \chi_A = \{ B \in \mathcal{A} \mid B \subseteq A \} \), and \( \mathcal{T}(A) = \{ A \in \mathcal{A} \} = \{ \emptyset \} \). We recall that \( \mathcal{A} \) is a base for the topology of \( \omega^* \), and that if \( A \in \mathcal{A} \), then \( A \) is homeomorphic to \( \omega^* \). Then, for every Banach space \( X \), \( C(A,X) \cong C(\omega^*,X) \). In what follows we often identify \( C(A,X) \) with the subspace \( \{ f : f_m^* = f \} \) of \( C(\omega^*,X) \). We also recall that the algebra \( \mathcal{A} \) has the following property (sometimes called Cantor separability): For every decreasing sequence \( (A_n) \) in \( \mathcal{A} \), there exists \( A \in \mathcal{A} \) which is contained in all \( A_n \). Finally, there is a result of Negrepontis that, under the CH, if \( A \) is an open \( F_\sigma \)-subset of \( \omega^* \), then its closure \( A \) is a retract of \( \omega^* \).

1. Lemma ([2]). Let \( \lambda : \mathcal{A} \to \mathbb{R} \) be a nondecreasing set function. Then for every \( A \in \mathcal{A} \) there exist \( B \in \mathcal{A}(A) \) and \( \beta \in \mathbb{R} \) such that

\[
\lambda(E) = \beta \quad \text{for all} \quad E \in \mathcal{T}(B).
\]

2. Theorem ([2]). If \( T : C(\omega^*) \to C(\omega^*) \) is an operator, then for every \( A \in \mathcal{A} \) there exists a \( B \in \mathcal{A}(A) \) and a scalar \( \gamma \) such that

\[
(Tf)|_B = \gamma f \quad \text{for all} \quad f \in C(B)
\]

As in [2], it will be convenient to prove this theorem in its equivalent form stated below. The proof presented here is somewhat different from that in [2], and we first give some explanations.
We recall that there is a one-to-one correspondence between the operators $T: C(\omega^*) \rightarrow C(\omega^*)$ and the bounded finitely additive vector measures $\mu : \mathcal{F} \rightarrow C(\omega^*)$. If $T$ is given, then the corresponding (representing) measure $\mu$ is defined by $\mu(E) = T(1_E)$; if $\mu$ is given, then the corresponding operator $T$ is defined by $Tf = \int f \, d\mu$.

Now suppose that $T$ and $\mu$ are related to each other in the above manner, and consider the conjugate operator $T^*: M(\omega^*) \rightarrow M(\omega^*)$, where $M(\omega^*)$ is the space of regular Borel measures on $\omega^*$ (identified with the dual of $C(\omega^*)$). For each $p \in \omega^*$ let $\mu_p = T^*\delta_p$, where $\delta_p$ is the Dirac measure at $p$. Then it is readily seen that

$$\mu(E)(p) = \mu_p(E) \quad \text{for all} \quad E \in \mathcal{F} \quad \text{and} \quad p \in \omega^*.$$ 

Let a measure $\nu \in M(\omega^*)$ be real-valued, and let $\nu^+$ be its positive part. Then $\nu^+$ is given for every Borel set $E \subset \omega^*$ by

$$\nu^+(E) = \sup_{B \subset E} \nu(B),$$

where the supremum is taken over all Borel subsets $B$ of $E$. Now, using regularity, it is easy to verify that

$$\nu^+(E) = \sup_{F \in \mathcal{F}} \nu(F) \quad \text{for all} \quad E \in \mathcal{C}.$$

In particular, for the real space $C(\omega^*)$, if $\mu : \mathcal{F} \rightarrow C(\omega^*)$ is a bounded measure, then

$$\mu^+_p(E) = \sup_{F \in \mathcal{F}} \mu_p(F) \quad \text{for all} \quad E \in \mathcal{F} \quad \text{and} \quad p \in \omega^*.$$ 

Hence, for every $E \in \mathcal{A}$, the function $p \mapsto \mu^+_p(E)$ is lower semi-continuous on $\omega^*$, and the same is of course true of the negative-part function $p \mapsto \mu^-_p(E) = (-\mu)^+_p(E)$. (The lower semicontinuity of the function $p \mapsto (T^*\delta_p)^+(E)$ holds in fact for every operator $T : C(K) \rightarrow C(K)$ and every open set $E \subset K$.)

Now we restate the above theorem in an equivalent form.

3. **Theorem ([2]).** Let $\mu : \mathcal{F} \rightarrow C(\omega^*)$ be a bounded finitely additive vector measure. Then for every $A \in \mathcal{F}_\omega$ there exist a $B$ in $\mathcal{F}_\omega(A)$ and a scalar $\gamma$ such that

$$\mu(E) \mathbb{1}_B = \gamma \mathbb{1}_A \quad \text{for all} \quad E \in \mathcal{F}(B).$$

**Proof.** We may (and will) assume that $C(\omega^*)$ is real. We start by defining two nondecreasing set functions $\lambda_\delta, \lambda_\omega : \mathcal{F}_\omega \rightarrow \mathbb{R}_+$ by

$$\lambda_\delta(E) = \sup_{p \in E} \mu^+_p(E) \quad \text{and} \quad \lambda_\omega(E) = \sup_{p \in E} \mu^-_p(E).$$
(It is easy to verify, using the formula for $v^*(E)$, $E \in \mathcal{L}$, given above, that these two functions coincide with those used in [2].) Let $A \in \mathcal{L}$. Applying Lemma 1 twice, we find $B \in \mathcal{L}(A)$ and $\alpha, \beta \in \mathbb{R}$ such that

$$\lambda_*(E) = \beta \text{ and } \lambda_*(E) = \alpha \text{ for all } E \in \mathcal{L}(B).$$

Let $E \in \mathcal{L}(B)$. If $F \in \mathcal{L}(E)$ and $p \in \omega^*$, then $\mu_*(F) \leq \mu_*(E)$. But

$$\sup_{p \in \omega} \mu_*(F) = \lambda_*(F) = \beta; \text{ so}$$

$$\sup_{p \in \omega} \mu_*(F) = \beta \text{ for all } F \in \mathcal{L}(E).$$

From this and the lower semicontinuity of the function $p \mapsto \mu_*(E)$ it follows that for every $\beta < \beta'$ the set $\{p \in E: \mu_*(E) > \beta'\}$ is open and dense in $E$. Hence the set

$$E^p = \{p \in E: \mu_*(E) = \beta\}$$

is a dense $G_\delta$-subset of $E$.

Next, if $E = B$, then

$$\beta = \lambda_*(B) = \mu_*(E) + \mu_*(B - E) = \mu_*(E) + \beta \text{ for all } p \in (B - E)^p$$

so that

$$\mu_*(E) = 0 \text{ for all } p \in (B - E)^p$$

But, by the lower semicontinuity again, the set $\{p \in B - E: \mu_*(E) = 0\}$ is closed in $B - E$, and it contains the set $(B - E)^p$ which is dense in $B - E$; therefore, $\mu_*(E) = 0$ for all $p \in B - E$. Thus

$$\mu_*(E) = \begin{cases} 
\beta & \text{for } p \in E^p, \\
0 & \text{for } p \in B - E.
\end{cases}$$

By a similar argument, the set

$$E_\alpha = \{p \in E: \mu_*(E) = \alpha\}$$

is a dense $G_\delta$-subset of $E$, and

$$\mu_*(E) = \begin{cases} 
\alpha & \text{for } p \in E^\alpha, \\
0 & \text{for } p \in B - E.
\end{cases}$$
Hence

$$\mu_\varepsilon(E) = \mu_\varepsilon'(E) - \mu_\varepsilon(E) = \begin{cases} \beta - \alpha = \gamma & \text{for } p \in E \cap E^\circ, \\ 0 & \text{for } p \in B - E. \end{cases}$$

But the function $\mu(E): p \mapsto \mu_\varepsilon(E)$ is continuous, and the set $E \cap E^\circ$ is dense in $E$, hence $\mu_\varepsilon(E) = \gamma$ for all $p \in E$.

We have thus shown that for every $E \in \mathcal{F}(B)$,

$$\mu_\varepsilon(E)(p) = \mu_\varepsilon(E) = \gamma 1_E(p) \text{ for all } p \in B,$$

which is precisely what was to be proved.\(\square\)

4. Corollary. $C(\omega^*)$ is a nice Banach space.

Proof. See [2], Proof of Corollary 2.4; see also Proof of Corollary 6 below.\(\square\)

Now we give an extension of Theorem 3 to the case of vector valued functions.

5. Theorem. Let $X$ be a separable Banach space, $Y$ a Banach space whose dual $Y^*$ is weak* separable, and let

$$T: C(\omega^*, X) \to C(\omega^*, Y)$$

be an operator. Then for every $A \in \mathcal{F}$, there exist $B \in \mathcal{F}(A)$ and $u \in L(X, Y)$ such that

$$(Tf)_1_a = uf \text{ for all } f \in C(B, X).$$

Proof. Let $(x_n)$ be a sequence dense in $X$, and $(y_n^*)$ a sequence in $Y^*$ separating the points of $Y$.

Given $x \in X$ and $y^* \in Y^*$, consider the bounded finitely additive measure

$$\mu_{x, y^*}: A \to \mu_{x, y^*}(A) = T(1_A x)$$

Then, by Theorem 3, for every $A \in \mathcal{F}$, there exists $B \in \mathcal{F}(A)$ and a scalar $\gamma$ such that

$$\mu_{x, y^*}(E) 1_B = \gamma 1_B \text{ for all } E \in \mathcal{F}(B).$$
Applying this inductively when \( y^* = y^*_n (n = 1, 2, \ldots) \) and \( x \) is held fixed, and then making use of the Cantor separability of \( \mathcal{L} \), we see that for every \( x \in X \) and \( A \in \mathcal{L} \), there exists a \( B \in \mathcal{L}(A) \) and a sequence of scalars \( (\gamma_n) \) such that

\[
\mu_{y^*_n}(E) 1_\phi = \gamma_n 1_\phi \quad \text{for all } E \in \mathcal{L}(B) \text{ and } n \in \mathbb{N}.
\]

Since the sequence \( (y^*_n) \) is total on \( Y \), it follows that there exists a (unique) \( y \in Y \) such that

\[
T(1_x) 1_\phi = 1_y \quad \forall \ E \in \mathcal{L}(B).
\]

Now, applying this inductively when \( x = x_m \) \((m = 1, 2, \ldots) \) and then using the Cantor separability of \( \mathcal{L} \) again, we find that for every \( A \in \mathcal{L} \), there exists a \( B \in \mathcal{L}(A) \) and a sequence \( (y_{m,n}) \) in \( Y \) such that

\[
T(1_{x_m}) 1_\phi = 1_{y_{m,n}} \quad \forall \ E \in \mathcal{L}(B) \text{ and } m \in \mathbb{N}.
\]

If \( x \in X \) and \( (x_m) \) is a subsequence of \( (x_n) \) converging to \( x \), then by the continuity of \( T \) there is a \( y = u(x) \in Y \) such that the sequence \( (y_{m,n}) \) converges to \( y \) (and this \( y \) does not depend on a particular choice of \( (x_n) \)). Thus

\[
T(1_{x_m}) 1_\phi = 1_{y_{m,n}} \quad \forall \ E \in \mathcal{L}(B) \text{ and } x \in X.
\]

Clearly, the mapping \( u : X \to Y \) is linear, and

\[
\|u(x)\| = \|1_x u(x)\|_{\infty} \leq \|T(1_x)\|_{\infty} \leq \|T\| \|x\| \quad \text{for all } x \in X
\]

so that \( u \in L(X,Y) \) (and \( \|u\| \leq \|T\| \)).

It follows that

\[
(Tf) 1_\phi = uf
\]

for every \( \omega \)-simple function \( f \) in \( C(B,X) \); since such functions are dense in \( C(B,X) \), the last equality holds for all \( f \) in \( C(B,X) \).

6. Corollary. If is a separable nice Banach space, then also the space \( C(\omega_*, X) \) is nice.

Proof. Let \( I \) denote the identity operator in \( C(\omega_*, X) \) and \( i \) the identity operator in \( X \). Let \( T \in L(C(\omega_*, X)) \). By Theorem 5, we can find \( B \in \mathcal{L} \) and \( u \in L(X) \) such that

\[
(Tf) 1_\phi = uf \quad \forall f \in C(B,X).
\]
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It is then easily checked that

\[(I-T)f \equiv (i-u)f \text{ for all } f \in C(B,X).\]

Since \(X\) is nice, there exists a subspace \(Y\) of \(X\) which is isomorphic to \(X\) and which is mapped isomorphically by \(u\) or \(i-u\) onto a complemented subspace of \(X\). Let's assume this holds for \(u\) so that \(u = id Y\) is an isomorphic embedding and \(u(Y) = u(Y) =: Z\) is complemented in \(X\). Let \(p\) be a projection from \(X\) onto \(Z\).

If \(f \in C(B,Y)\), then \((Tf)_* = vof\) and so

\[\|v^{-1}\|^{-1} \|f\|_* \leq \|vof\|_* \leq \|Tf\|_* \leq \|T\|_* \|f\|_*\]

which shows that \(T : C(B,Y)\) is an isomorphic embedding of \(C(B,Y)\) into \(C(w^*,X)\). Define an operator \(P : C(w^*,X) \to C(w^*,X)\) by

\[Pf = T(v^{-1}p * g1)_*\]

Clearly, the range of \(P\) is contained in \(T[C(B,Y)]\). If \(g \in T[C(B,Y)]\), i.e., \(g = Tf\) for some \(f \in C(B,Y)\), then \(g1_* = (Tf)_* = vof\) and hence \(Pf = T(v^{-1}p * vof) = T(v^{-1} * vof) = Tf = g\). Thus \(P\) is a projection from \(C(w^*,X)\) onto its subspace \(T[C(B,Y)] = C(B,Y) = C(B,X) = C(w^*,X)\).

As easily seen, for every compact space \(K\) and every Banach space \(X\), there is a natural isometric isomorphism between the spaces \(C(K,c_c(X))\) and \(c_c(C(K,X))\) so that

\[C(K,c_c(X)) \cong c_c(C(K,X))\]

We use this fact in our next result.

7. Corollary. If \(X\) is a separable nice Banach space which is isomorphic to its \(c_c\)-sum \(c_c(X)\), then the space \(C(w^*,X)\) is primary.

Proof. We first observe that

\[C(w^*,X) = C(w^*,c_c(X)) = c_c(C(w^*,X));\]

thus, denoting shortly \(C(w^*,X) := C\), we have \(C \approx c_c(C)\).

Now let \(C = E \oplus F\). By Corollary 6, one of the summands, \(E\) say, contains a complemented subspace \(V\) which is isomorphic to \(C\). Thus there is a subspace \(U\) in \(E\) such that
Applying Pelczynski's decomposition method, we now get:

\[ E = U \oplus V, \text{ where } V = C\sim c_0(C). \]

In particular, we have the following.

8. Corollary. For every infinite metrizable compact space \( K \), the space \( C(\omega^*, C(K)) \equiv C(\omega^* \times K) \) is primary.

Proof. This follows directly from the preceding corollary because such spaces \( C(K) \) are known to be nice ([3], [1]) and isomorphic with their \( c_0 \)-sums [4].

9. Remark. Let \( X \) be an arbitrary Banach space. Define \( \kappa(X) \) to be the Banach space of all relatively norm compact sequences \( (x_n) \) in \( X \), endowed with the supremum norm. Then

\[ \kappa(X)/c_0(X) \cong C(\omega^*, X). \]

This can be verified precisely as in the scalar case, using the Stone-Čech isometric isomorphism between \( \kappa(X) \) and \( C(\beta\omega, X) \), and the fact (surely well known) that Tietze's type extensions from \( \omega^* \) to \( \beta\omega \) exist for continuous \( X \)-valued functions. For the sake of completeness, we give a sketch of that fact:

Let \( g \in C(\omega^*, X) \). Then there exists a sequence \((g_n)\) of \( \omega \)-simple functions in \( C(\omega^*, X) \) converging uniformly to \( g \). For each \( n \), choose a finite \( \omega \)-partition \( \omega^*_n = \{A^*_n,...,A^*_{n}\} \) so that \( g_n \) assumes constant (not necessarily distinct) values on each of the sets \( A^*_n \); this can be done so that \( \omega^*_n \) is a refinement of \( \omega^*_n \). Then it is easily seen that we can define a sequence of partitions of \( \omega^*_n \) consisting of \( \omega \)-sets and such that \( \omega^*_n \) is a refinement of \( \omega^*_n \) and that \( \omega^*_n \) consists of \( \omega^*_n \)-sets. Let \( \omega^* \in \kappa(X) \) be the sequence which takes the constant value \( x_n \) on the set \( M^*_n \), where \( \{x_n\} = g(A^*_n), i = 1,...,k_n \). Finally, \( f_n \) be the continuous extension of \( x^* \) to \( \beta\omega \). Then \( f_n(\omega^*) = g_n \) and \( \|f_n - f_k\| = \|g_n - g_k\| \) for all \( m \) and \( n \) so that the sequence \((f_n)\) converges uniformly to a function \( f \in C(\beta\omega, X), f(\omega^*) = g \) and \( \|f\| = \|g\|_\omega \).

10. Remark. For a compact space \( K \) and a Banach space \( X \), let \( l_\infty(C(K, X)) \) denote the Banach space consisting of all sequences \((f_n)\) such that \( f_n \in C(K, X) \) for each \( n \) and the joint range of the functions \( f_n \) that is, \( \cup_{n \in \mathbb{N}} f_n(K) \), is a relatively norm compact subset of \( X \), with the norm defined by \( \|l(f_n)\| = \sup_{n \in \mathbb{N}} \|f_n\|_\omega \). Then the same argument as in the proof of Proposition 3.2 in [2] shows that, under the CH (which enters here via the result of Negrepontis mentioned before Lemma 1), \( l_\infty(C(\omega^*, X)) \) is isometric to a complemented sub-
space of $C(\omega^*,\mathcal{X})$ from which, as a consequence, we have that $l_1(C(\omega^*,\mathcal{X})) = C(\omega^*,\mathcal{X})$. Unfortunately, we cannot apply this result to the primariness problem of the spaces $C(\omega^*,\mathcal{X})$ because we do not know if any analog of the fact that $l_n(E \oplus F) = l_n(E) \oplus l_n(F)$ holds for our $l_n$-sums.

References


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