The SL(2, C)-Character Varieties of Torus Knots

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ABSTRACT

Let G be the fundamental group of the complement of the torus knot of type (m, n). This has a presentation $G = \langle x, y \mid x^m = y^n \rangle$. We find the geometric description of the character variety $X(G)$ of characters of representations of $G$ into SL(2, C).

Key words: Torus knot, characters, representations.


Introduction

Since the foundational work of Culler and Shalen [1], the varieties of SL(2, C)-characters have been extensively studied. Given a manifold $M$, the variety of representations of $\pi_1(M)$ into SL(2, C) and the variety of characters of such representations both contain information of the topology of $M$. This is specially interesting for 3-dimensional manifolds, where the fundamental group and the geometrical properties of the manifold are strongly related.

This can be used to study knots $K \subset S^3$, by analysing the SL(2, C)-character variety of the fundamental group of the knot complement $S^3 - K$. In this paper, we study the case of the torus knots $K_{m,n}$ of any type $(m, n)$. The case $(m, n) = (m, 2)$ was analysed in [3] and the general case was recently determined in [2] by a method different from ours.

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1. Character varieties

A representation of a group $G$ in $\text{SL}(2, \mathbb{C})$ is a homomorphism $\rho : G \to \text{SL}(2, \mathbb{C})$. Consider a finitely presented group $G = \langle x_1, \ldots, x_k | r_1, \ldots, r_s \rangle$, and let $\rho : G \to \text{SL}(2, \mathbb{C})$ be a representation. Then $\rho$ is completely determined by the $k$-uple $(A_1, \ldots, A_k) = (\rho(x_1), \ldots, \rho(x_k))$ subject to the relations $r_j(A_1, \ldots, A_k) = 0$, $1 \leq j \leq s$. Using the natural embedding $\text{SL}(2, \mathbb{C}) \subset \mathbb{C}^4$, we can identify the space of representations as

$$R(G) = \text{Hom}(G, \text{SL}(2, \mathbb{C})) = \{(A_1, \ldots, A_k) \in \text{SL}(2, \mathbb{C})^k \mid r_j(A_1, \ldots, A_k) = 0, 1 \leq j \leq s\} \subset \mathbb{C}^4.$$

Therefore $R(G)$ is an affine algebraic set.

We say that two representations $\rho$ and $\rho'$ are equivalent if there exists $P \in \text{SL}(2, \mathbb{C})$ such that $\rho'(g) = P^{-1} \rho(g) P$, for every $g \in G$. This produces an action of $\text{SL}(2, \mathbb{C})$ in $R(G)$. The moduli space of representations is the GIT quotient

$$M(G) = \text{Hom}(G, \text{SL}(2, \mathbb{C}))/\text{SL}(2, \mathbb{C}).$$

A representation $\rho$ is reducible if the elements of $\rho(G)$ all share a common eigenvector, otherwise $\rho$ is irreducible.

Given a representation $\rho : G \to \text{SL}(2, \mathbb{C})$, we define its character as the map $\chi_\rho : G \to \mathbb{C}$, $\chi_\rho(g) = \text{tr} \rho(g)$. Note that two equivalent representations $\rho$ and $\rho'$ have the same character, and the converse is also true if $\rho$ or $\rho'$ is irreducible [1, Proposition 1.5.2].

There is a character map $\chi : R(G) \to \mathbb{C}^2$, $\rho \mapsto \chi_\rho$, whose image

$$X(G) = \chi(R(G))$$

is called the character variety of $G$. Let us give $X(G)$ the structure of an algebraic variety. By the results of [1], there exists a collection $g_1, \ldots, g_a$ of elements of $G$ such that $\chi_\rho$ is determined by $\chi_\rho(g_1), \ldots, \chi_\rho(g_a)$, for any $\rho$. Such collection gives a map

$$\Psi : R(G) \to \mathbb{C}^a, \quad \Psi(\rho) = (\chi_\rho(g_1), \ldots, \chi_\rho(g_a)).$$

We have a bijection $X(G) \cong \Psi(R(G))$. This endows $X(G)$ with the structure of an algebraic variety. Moreover, this is independent of the chosen collection as proved in [1].

**Lemma 1.1.** The natural algebraic map $M(G) \to X(G)$ is a bijection.

**Proof.** The map $R(G) \to X(G)$ is algebraic and $\text{SL}(2, \mathbb{C})$-invariant, hence it descends to an algebraic map $\varphi : M(G) \to X(G)$. Let us see that $\varphi$ is a bijection.

For $\rho$ an irreducible representation, if $\varphi(\rho) = \varphi(\rho')$ then $\rho$ and $\rho'$ are equivalent representations; so they represent the same point in $M(G)$.

Now suppose that $\rho$ is reducible. Consider $e_1 \in \mathbb{C}^2$ the common eigenvector of all $\rho(g)$. This gives a sub-representation $\rho' : G \to \mathbb{C}^*$ of $G$. We have a quotient
representation $\rho'' = \rho / \rho^\prime : G \to \mathbb{C}^*$, defined as the representation induced by $\rho$ in the quotient space $\mathbb{C}^2 / \langle \varepsilon_1 \rangle$. As characters, $\rho'' = \rho^\prime - 1$. The representation $\rho'' = \rho' \oplus \rho''$ is the semisimplification of $\rho$. It is in the closure of the SL(2, $\mathbb{C}$)-orbit through $\rho$. Clearly, $\chi_{\rho'}(g) = \rho'(g) + \rho'(g)^{-1}$. Now if $\rho$ and $\tilde{\rho}$ are two reducible representations and $\varphi(\rho) = \varphi(\tilde{\rho})$, then their semisimplifications have the same character, that is $\chi_{\rho'}(g) = \chi_{\tilde{\rho}'}(g)$. Therefore $\rho' = \tilde{\rho}'$ or $\rho' = \tilde{\rho}' - 1$. In either case $\rho$ and $\tilde{\rho}$ represent the same point in $M(G)$, which is actually the point represented by $\rho' \oplus \rho'^{-1}$. \hfill $\square$

2. Character varieties of torus knots

Let $T^2 = S^1 \times S^1$ be the 2-torus and consider the standard embedding $T^2 \subset S^3$. Let $m, n$ be a pair of coprime positive integers. Identifying $T^2$ with the quotient $\mathbb{R}^2 / \mathbb{Z}^2$, the image of the straight line $y = \frac{m}{n} x$ in $T^2$ defines the torus knot of type $(m, n)$, which we shall denote as $K_{m,n} \subset S^3$ (see [4, Chapter 3]).

For any knot $K \subset S^3$, we denote by $G(K)$ the fundamental group of the exterior $S^3 - K$ of the knot. It is known that $G_{m,n} = G(K_{m,n}) \cong \langle x, y \mid x^m = y^n \rangle$.

The purpose of this paper is to describe the character variety $X(G_{m,n})$.

In [3], the character variety $X(G_{m,2})$ is computed. We want to extend the result to arbitrary $m, n$, and give a simpler argument than that of [3].

After the completion of this work, we became aware of the paper [2] where the character varieties of $X(G_{m,n})$ are determined (even without the assumption of $m, n$ being coprime). However, our method is more direct than the one presented in [2].

To start with, note that $R(G_{m,n}) = \{ (A, B) \in \text{SL}(2, \mathbb{C}) \mid A^m = B^n \}$. Therefore we shall identify a representation $\rho$ with a pair of matrices $(A, B)$ satisfying the required relation $A^m = B^n$.

We decompose the character variety $X(G_{m,n}) = X_{\text{red}} \cup X_{\text{irr}}$, where $X_{\text{red}}$ is the subset consisting of the characters of reducible representations (which is a closed subset by [1]), and $X_{\text{irr}}$ is the closure of the subset consisting of the characters of irreducible representations.

**Proposition 2.1.** There is an isomorphism $X_{\text{red}} \cong \mathbb{C}$. The correspondence is defined by $\rho = \begin{pmatrix} A & \begin{pmatrix} t^n & 0 \\ 0 & t^{-n} \end{pmatrix} \\ B & \begin{pmatrix} t^m & 0 \\ 0 & t^{-m} \end{pmatrix} \end{pmatrix} \mapsto s = t + t^{-1} \in \mathbb{C}$. 

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Proof. By the discussion in Lemma 1.1, an element in $X_{\text{red}}$ is described as the character of a split representations $\rho = \rho' \oplus \rho'^{-1}$. This means that in a suitable basis,

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix},$$

The equality $A^m = B^n$ implies $\lambda^m = \mu^n$. Therefore there is a unique $t \in \mathbb{C}$ with $t \neq 0$ such that

$$\begin{cases} 
\lambda = t^n, \\
\mu = t^m.
\end{cases}$$

(Here we use the coprimality of $(m, n)$). Note that the pair $(A, B)$ is well-defined up to permuting the two vectors in the basis. This corresponds to the change $(\lambda, \mu) \mapsto (\lambda^{-1}, \mu^{-1})$, which in turn corresponds to $t \mapsto t^{-1}$. So $(A, B)$ is parametrized by $s = t + t^{-1} \in \mathbb{C}$. \qed

Lemma 2.2. Suppose that $\rho = (A, B) \in R(G_{m,n})$. In any of the following cases:

(a) $A^m = B^n \neq \pm \text{Id}$,

(b) $A = \pm \text{Id}$ or $B = \pm \text{Id}$,

(c) $A$ or $B$ is non-diagonalizable,

the representation $\rho$ is reducible.

Proof. First suppose that $A$ is diagonalizable with eigenvalues $\lambda, \lambda^{-1}$, and suppose that $\lambda^m \neq \pm 1$. Then there is a basis $e_1, e_2$ in which $A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, which is well-determined up to multiplication of the basis vectors by non-zero scalars. Then

$$B^n = A^m = \begin{pmatrix} \lambda^m & 0 \\ 0 & \lambda^{-m} \end{pmatrix}$$

is a diagonal matrix, different from $\pm \text{Id}$. Therefore $B$ must be diagonal in the same basis, $B = \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix}$, with $\lambda^m = \mu^n$. This proves the reducibility in case (a).

Now suppose that $A = \lambda \text{Id}$, $\lambda = \pm 1$. Then $B^n = \lambda^m \text{Id}$, so it must be that $B$ is diagonalizable. Using a basis in which $B$ is diagonal, we get the reducibility in case (b).

Finally, suppose that $A$ is not diagonalizable. Then there is a suitable basis on which $A$ takes the form $A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$, with $\lambda = \pm 1$. Clearly

$$B^n = A^m = \lambda^m \begin{pmatrix} 1 & m\lambda \\ 0 & 1 \end{pmatrix}$$
Proof. Let 

$$B = \begin{pmatrix} \mu & x \\ 0 & \mu \end{pmatrix},$$

with \(\mu = \pm 1\), \(\mu^n = \lambda^m\) and \(\mu^n x = \lambda m\). In this basis, the vector \(e_1\) is an eigenvector for both \(A\) and \(B\). Hence the representation \((A, B)\) is reducible, completing the case (c).

**Proposition 2.3.** Let \(X_{irr}^o\) be the set of irreducible characters, and \(X_{irr}\) its closure. Then

$$\begin{align*}
X_{irr}^o & \cong \{(\lambda, \mu, r) \mid \lambda^m = \mu^n = \pm 1, \lambda \neq \pm 1, \mu \neq \pm 1, r \in \mathbb{C} - \{0, 1\}\}/\mathbb{Z}_2 \times \mathbb{Z}_2, \\
X_{irr} & \cong \{(\lambda, \mu, r) \mid \lambda^m = \mu^n = \pm 1, \lambda \neq \pm 1, \mu \neq \pm 1, r \in \mathbb{C}\}/\mathbb{Z}_2 \times \mathbb{Z}_2,
\end{align*}$$

where \(\mathbb{Z}_2 \times \mathbb{Z}_2\) acts as \((\lambda, \mu, r) \sim (\lambda^{-1}, \mu, 1 - r) \sim (\lambda, \mu^{-1}, 1 - r) \sim (\lambda^{-1}, \mu^{-1}, r)\).

**Proof.** Let \(\rho = (A, B)\) be an element of \(R(G_m, n)\) which is an irreducible representation. By Lemma 2.2, \(A\) is diagonalizable but not equal to \(\pm \text{Id}\), and \(A^n = \pm \text{Id}\). So the eigenvalues \(\lambda, \lambda^{-1}\) of \(A\) satisfy \(\lambda^m = \pm 1\) and \(\lambda \neq \pm 1\). Analogously, \(B\) is diagonalizable but not equal to \(\pm \text{Id}\), with eigenvalues \(\mu, \mu^{-1}\), with \(\mu^n = \pm 1, \mu \neq \pm 1\). Moreover,

$$\lambda^m = \mu^n.$$

We may choose a basis \(\{e_1, e_2\}\) under which \(A\) diagonalizes. This is well-defined up to multiplication of \(e_1\) and \(e_2\) by two non-zero scalars. Let \(\{f_1, f_2\}\) be a basis under which \(B\) diagonalizes, which is well-defined up to multiplication of \(f_1, f_2\) by non-zero scalars. Then \(\{[e_1], [e_2], [f_1], [f_2]\}\) are four points of the projective line \(\mathbb{P}^1 = \mathbb{P}(\mathbb{C}^2)\).

Note that the pair \((A, B)\) is irreducible if and only if the four points are different.

The only invariant of four points in \(\mathbb{P}^1\) is the double ratio

$$r = ([e_1] : [e_2] : [f_1] : [f_2]) \in \mathbb{P}^1 - \{0, 1, \infty\} = \mathbb{C} - \{0, 1\}.$$ 

So \((A, B)\) is parametrized, up to the action of \(\text{SL}(2, \mathbb{C})\), by \((\lambda, \mu, r)\). Permuting the two basis vectors \(e_1, e_2\) corresponds to \((\lambda, \mu, r) \mapsto (\lambda^{-1}, \mu, 1 - r)\), since

$$([e_2] : [e_1] : [f_1] : [f_2]) = 1 - ([e_1] : [e_2] : [f_1] : [f_2]).$$

Analogously, permuting the two basis vectors \(f_1, f_2\) corresponds to

$$(\lambda, \mu, r) \mapsto (\lambda^{-1}, 1 - r).$$

Note that this gives an action of \(\mathbb{Z}_2 \times \mathbb{Z}_2\) and \(X_{irr}^o\) is the quotient of the set of \((\lambda, \mu, r)\) as above by this action.

To describe the closure of \(X_{irr}^o\), we have to allow \(f_1\) to coincide with \(e_1\). This corresponds to \(r = 1\) (the same happens if \(f_2\) coincides with \(e_2\)). In this case, \(e_1\) is
an eigenvector of both $A$ and $B$, so the representation $(A, B)$ has the same character as its semisimplification $(A', B')$ given by

$$A' = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \quad B' = \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix}. $$

This means that the point $(\lambda, \mu, 1)$ corresponds under the identification $X_{\text{red}} \cong \mathbb{C}$ given by Proposition 2.1 to $s_1 = t_1 + t_1^{-1}$, where $t_1 \in \mathbb{C}$ satisfies

$$\begin{cases} 
\lambda = t_1^n, \\
\mu = t_1^m.
\end{cases}$$

(1)

Also, we have to allow $f_1$ to coincide with $e_2$ (or $f_2$ to coincide with $e_1$). This corresponds to $r = 0$. The representation $(A, B)$ has semisimplification $(A', B')$ where

$$A' = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \quad B' = \begin{pmatrix} \mu^{-1} & 0 \\ 0 & \mu \end{pmatrix}. $$

So the point $(\lambda, \mu, 1)$ corresponds to $s_0 = t_0 + t_0^{-1} \in X_{\text{red}} \cong \mathbb{C}$, where $t_0 \in \mathbb{C}$ satisfies

$$\begin{cases} 
\lambda = t_0^n, \\
\mu^{-1} = t_0^m.
\end{cases}$$

(2)

Proposition 2.3 says that $X_{\text{irr}}$ is a collection of $\frac{(m-1)(n-1)}{2}$ lines. A pair $(\lambda, \mu)$ with $\lambda^n = \pm 1$ and $\mu^n = \pm 1$ is given as

$$\lambda = e^{\pi ik/m}, \quad \mu = e^{\pi i k'/n},$$

where $0 \leq k < 2m$, $0 \leq k' < 2n$. The condition $\lambda \neq \pm 1, \mu \neq \pm 1$ gives $k \neq 0, m$, $k' \neq 0, n$. Finally, the $\mathbb{Z}_2 \times \mathbb{Z}_2$-action allows us to restrict to $0 < k < m$, $0 < k' < n$. The condition $\lambda^n = \mu^n$ means that

$$k \equiv k' \pmod{2}.$$  

Denote by $X_{\text{irr}}^{k,k'}$ the line of $X_{\text{irr}}$ corresponding to the values of $k, k'$. Then

$$X_{\text{irr}} = \bigcup_{0 < k < m, \ 0 < k' < n, \ k \equiv k' \pmod{2}} X_{\text{irr}}^{k,k'}.$$  

The line $X_{\text{irr}}^{k,k'}$ intersects $X_{\text{red}}$ in two points. This gives a collection of $(m-1)(n-1)$ points in $X_{\text{red}}$, which are defined as follows: under the identification $X_{\text{red}} \cong \mathbb{C}$, these are the points $s_1 = t_1 + t_1^{-1}$, where

$$t_1 = e^{\pi i / nm},$$
and $0 < l < mn, m \nmid l, n \nmid l$. Assume that $n$ is odd (note that either $m$ or $n$ should be odd). Then from (1) and (2), the line $X_{irr}^{k,k'}$ intersects at the points $s_{l_0}, s_{l_1} \in X_{red}$ where

$$nl_0 \equiv k \pmod{m}, \quad ml_0 \equiv n - k' \pmod{n},$$

$$nl_1 \equiv k \pmod{m}, \quad ml_1 \equiv k' \pmod{n}.$$  

These two points are different since $k' \not\equiv n - k' \pmod{n}$, as $n$ is odd.

In the case $(m, n) = (2, n)$, this result coincides with [3, Corollary 4.2].

### 3. The algebraic structure of $X(G_{m,n})$

We want to give a geometric realization of $X(G_{m,n})$ which shows that the algebraic structure of this variety is that of a collection of rational lines as in Figure 1 intersecting with nodal curve singularities.

The map $R(G_{m,n}) \to \mathbb{C}^3, \rho = (A, B) \mapsto (\text{tr}(A), \text{tr}(B), \text{tr}(AB))$, defines a map

$$\Psi: X(G_{m,n}) \to \mathbb{C}^3.$$

**Theorem 3.1.** The map $\Psi$ is an isomorphism with its image $C = \Psi(X(G_{m,n}))$. $C$ is a curve consisting of $\frac{(n-1)(m-1)}{2} + 1$ irreducible components, all of them smooth and isomorphic to $C$. They intersect with nodal normal crossing singularities following the pattern in Figure 1.
Proof. Let us look first at $\Psi_0 = \Psi|_{X_{\text{red}}} : X_{\text{red}} \to \mathbb{C}^3$. For a given $\rho = (A,B) \in X_{\text{red}}$, with the shape given in Proposition 2.1, we have that

$$\Psi_0 : s = t + t^{-1} \mapsto (t^n + t^{-n}, t^m + t^{-m}, t^{n+m} + t^{-(n+m)}).$$

This map is clearly injective: the image recovers

$$\{t^n, t^{-n}\}, \{t^m, t^{-m}\}, \{t^{n+m}, t^{-(n+m)}\}.$$ 

From this, we recover $\{(t^n, t^m), (t^{-n}, t^{-m})\}$ and hence the pair $t, t^{-1}$ (since $n,m$ are coprime).

Let us see that $\Psi_0$ is an immersion. The differential is

$$\frac{d\Psi_0}{dt} = \left(mt^{-n-1}(t^{2n} - 1), mt^{-m-1}(t^{2m} - 1), (n + m)t^{-n-m-1}(t^{2n+2m} - 1)\right).$$

(3)

This is non-zero at all $t \neq \pm 1$. As $\frac{d\Psi_0}{ds} \neq (0,0,0)$, for $t = \pm 1$, we note that $\frac{d\Psi_0}{dt} = t^{-2}(t^2 - 1)$, so

$$\frac{d\Psi_0}{ds} = \left(nt^{-n+1}(t^{2n} - 1), nt^{-m+1}(t^{2m} - 1), (n + m)t^{-n+m+1}(t^{2n+2m} - 1)\right),$$

which is non-zero again.

Now, consider a component of $X_{\mu,r}$ corresponding to a pair $(\lambda, \mu)$. Take $r \in \mathbb{C}$. Fix the basis $\{e_1, e_2\}$ of $\mathbb{C}^2$ which is given as the eigenbasis of $A$. Let $\{f_1, f_2\}$ be the eigenbasis of $B$. As the double ratio $0 : \infty : 1 : r/(r-1)) = r$, we can take $f_1 = (1,1)$ and $f_2 = (r-1, r)$. This corresponds to the matrices:

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix},$$

$$B = \begin{pmatrix} 1 & r-1 \\ 1 & r \end{pmatrix} \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix} \begin{pmatrix} 1 & r-1 \\ 1 & r \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} r(\mu - \mu^{-1}) + \mu^{-1} & (1-r)(\mu - \mu^{-1}) \\ r(\mu - \mu^{-1}) & \mu - r(\mu - \mu^{-1}) \end{pmatrix}.$$ 

Therefore:

$$\Psi(A, B) = (\text{tr}(A), \text{tr}(B), \text{tr}(AB))$$

$$= (\lambda + \lambda^{-1}, \mu + \mu^{-1} + \mu(\lambda + \mu^{-1} + \lambda^{-1}), (\mu(\lambda - \lambda^{-1}) + (\mu - \mu^{-1})).$$

The image of this component is a line in $\mathbb{C}^3$. Its direction vector is $(0,0,1)$. At an intersection point with $\Psi_0(X_{\text{red}})$, the tangent vector to $\Psi_0(X_{\text{red}})$, given in (3), has non-zero first and second component, since $\lambda = t^n$, $\mu = t^m$ and $t \neq 0$, $\lambda^2 \neq 1$, $\mu^2 \neq 1$. So the intersection of these components is a transverse nodal singularity.

Finally, note that the map $\Psi : X(G_{m,n}) \to C$ is an algebraic map, it is a bijection, and $C$ is a nodal curve (the mildest possible type of singularities). Therefore $\Psi$ must be an isomorphism.

\[\square\]
Corollary 3.2. $M(G) \cong X(G)$, for $G = G_{m,n}$.

Proof. By Lemma 1.1, $\varphi : M(G) \to X(G)$ is an algebraic map which is a bijection. As the singularities of $X(G)$ are just transverse nodes, $\varphi$ must be an isomorphism. □

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References


