Tempered Radon Measures

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ABSTRACT

A tempered Radon measure is a $\sigma$-finite Radon measure in $\mathbb{R}^n$ which generates a tempered distribution. We prove the following assertions. A Radon measure $\mu$ is tempered if, and only if, there is a real number $\beta$ such that $(1 + |x|^2)^{\frac{\beta}{2}} \mu$ is finite. A Radon measure is finite if, and only if, it belongs to the positive cone $\tilde{B}_{1,\infty}^0(\mathbb{R}^n)$ of $B_{1,\infty}^0(\mathbb{R}^n)$. Then $\mu(\mathbb{R}^n) \sim \| \mu \|_{B_{1,\infty}^0(\mathbb{R}^n)}$ (equivalent norms).

Key words: Radon measure, tempered distributions, Besov spaces

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Introduction

A substantial part of fractal geometry and fractal analysis deals with Radon measures in $\mathbb{R}^n$ (also called fractal measures) with compact support. One may consult [5] and the references given there. In the present paper we clarify the relation between arbitrary $\sigma$-finite Radon measure in $\mathbb{R}^n$, tempered distributions and weighted Besov spaces. It comes out that a $\sigma$-finite Radon measure $\mu$ in $\mathbb{R}^n$ can be identified with a tempered distribution $\mu \in S'(\mathbb{R}^n)$ if and only if there is a real number $\beta$ such that

$$\mu_{\beta}(\mathbb{R}^n) < \infty, \quad \text{where} \quad \mu_{\beta} = (1 + |x|^2)^{\frac{\beta}{2}} \mu.$$

Radon measures $\mu$ with $\mu(\mathbb{R}^n) < \infty$ are called finite. These finite Radon measures can be identified with the positive cone $\tilde{B}_{1,\infty}^0(\mathbb{R}^n)$ of the distinguished Besov space $B_{1,\infty}^0(\mathbb{R}^n)$ and

$$\| \mu \|_{B_{1,\infty}^0(\mathbb{R}^n)} \sim \mu(\mathbb{R}^n)$$
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(equivalent norms).

This paper is organised as follows. In section 1 we collect the definitions and preliminaries. We introduce the well-known weighted Besov spaces \( B_{pq}^s(\mathbb{R}^n, \langle x \rangle^\alpha) \) and prove that for fixed \( p, q \) with \( 0 < p, q \leq \infty \)

\[
S(\mathbb{R}^n) = \bigcap_{\alpha, s \in \mathbb{R}} B_{pq}^s(\mathbb{R}^n, \langle x \rangle^\alpha)
\]

and

\[
S'(\mathbb{R}^n) = \bigcup_{\alpha, s \in \mathbb{R}} B_{pq}^s(\mathbb{R}^n, \langle x \rangle^\alpha).
\]

Although known to specialists we could not find an explicit reference. In section 2 we prove in the Theorems 2.1 and 2.2 the above indicated main results.

1. Definitions and preliminaries

Let \( \mathbb{N} \) be the collection of all natural numbers and \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \). Let \( \mathbb{R}^n \) be Euclidean \( n \)-space, where \( n \in \mathbb{N} \). Put \( \mathbb{R} = \mathbb{R}^1 \), whereas \( \mathbb{C} \) is the complex plane. Let \( S(\mathbb{R}^n) \) be the Schwartz space of all complex-valued, rapidly decreasing, infinitely differentiable functions on \( \mathbb{R}^n \). By \( S'(\mathbb{R}^n) \) we denote its topological dual, the space of all tempered distributions on \( \mathbb{R}^n \). \( L_p(\mathbb{R}^n) \) with \( 0 < p \leq \infty \) is the standard quasi-Banach space with respect to Lebesgue measure, quasi-normed by

\[
\| f \|_{L_p(\mathbb{R}^n)} = \left( \int_{\mathbb{R}^n} |f(x)|^p \, dx \right)^{1/p}, \quad 0 < p < \infty
\]

with the standard modification if \( p = \infty \).

If \( \varphi \in S(\mathbb{R}^n) \) then

\[
\hat{\varphi}(\xi) = F\varphi(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \varphi(x) e^{-ix\xi} \, dx, \quad \xi \in \mathbb{R}^n,
\]

denotes the Fourier transform of \( \varphi \). The inverse Fourier transform is given by

\[
\hat{\varphi}(x) = F^{-1}\varphi(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \varphi(\xi) e^{ix\xi} \, d\xi, \quad x \in \mathbb{R}^n.
\]

One extends \( F \) and \( F^{-1} \) in the usual way from \( S \) to \( S' \). For \( f \in S'(\mathbb{R}^n) \),

\[
Ff(\varphi) = f(F\varphi), \quad \varphi \in S(\mathbb{R}^n).
\]

Let \( \varphi_0 \in S(\mathbb{R}^n) \) with

\[
\varphi_0(x) = 1, \quad |x| \leq 1 \quad \text{and} \quad \varphi_0(x) = 0, \quad |x| \geq \frac{3}{2}, \quad (1)
\]

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and let
\[ \varphi_k(x) = \varphi_0(2^{-k}x) - \varphi_0(2^{-k+1}x), \quad x \in \mathbb{R}^n, \quad k \in \mathbb{N}. \] (2)

Then, since
\[ 1 = \sum_{j=0}^{\infty} \varphi_j(x) \quad \text{for all} \quad x \in \mathbb{R}^n, \] (3)

the \( \varphi_j \) form a dyadic resolution of unity in \( \mathbb{R}^n \). \( (\varphi_k \hat{f})^{-} \) is an entire analytic function on \( \mathbb{R}^n \) for any \( f \in S'(\mathbb{R}^n) \). In particular, \( (\varphi_k \hat{f})^{-}(x) \) makes sense pointwise.

**Definition 1.1.** Let \( \varphi = \{ \varphi_j \}_{j=0}^{\infty} \) be the dyadic resolution of unity according to (1)–(3), \( s \in \mathbb{R}, \ 0 < p \leq \infty, \ 0 < q \leq \infty \), and

\[ \| f \|_{B_{pq}^s(\mathbb{R}^n)} = \left( \sum_{j=0}^{\infty} 2^{jsq} \| (\varphi_k \hat{f})^{-} \|_{L_p(\mathbb{R}^n)}^q \right)^{\frac{1}{q}} \] (with the usual modification if \( q = \infty \)). Then the Besov space \( B_{pq}^s(\mathbb{R}^n) \) consists of all \( f \in S'(\mathbb{R}^n) \) such that \( \| f \|_{B_{pq}^s(\mathbb{R}^n)} < \infty \).

We denote by \( L_p(\mathbb{R}^n, \langle x \rangle^\alpha) \), where
\[ \langle x \rangle^\alpha = (1 + |x|^2)^{\frac{\alpha}{2}}, \] the weighted \( L_p \)-space quasi-normed by
\[ \| f \|_{L_p(\mathbb{R}^n, \langle x \rangle^\alpha)} = \| \langle \cdot \rangle^\alpha f \|_{L_p(\mathbb{R}^n)}. \]

**Definition 1.2.** Let \( \varphi = \{ \varphi_j \}_{j=0}^{\infty} \) be the dyadic resolution of unity according to (1)–(3), \( s \in \mathbb{R}, \ 0 < p \leq \infty, \ 0 < q \leq \infty \). Then the weighted Besov space \( B_{pq}^s(\mathbb{R}^n, \langle x \rangle^\alpha) \) is a collection of all \( f \in S'(\mathbb{R}^n) \) such that

\[ \| f \|_{B_{pq}^s(\mathbb{R}^n, \langle x \rangle^\alpha)} = \left( \sum_{j=0}^{\infty} 2^{jsq} \| (\varphi_k \hat{f})^{-} \|_{L_p(\mathbb{R}^n, \langle x \rangle^\alpha)}^q \right)^{\frac{1}{q}} \] (with the usual modification if \( q = \infty \)) is finite.

**Remark 1.3.** If \( \alpha = 0 \) then we have the space \( B_{pq}^s(\mathbb{R}^n) \) as introduced in Definition 1.1. It is also known from [1, ch. 4.2.2] that the operator \( f \mapsto \langle x \rangle^\alpha f \) is an isomorphic mapping from \( B_{pq}^s(\mathbb{R}^n, \langle x \rangle^\alpha) \) onto \( B_{pq}^s(\mathbb{R}^n) \). In particular,
\[ \| \langle \cdot \rangle^\alpha f \|_{B_{pq}^s(\mathbb{R}^n)} \sim \| f \|_{B_{pq}^s(\mathbb{R}^n, \langle x \rangle^\alpha)}. \]

Next we review some special properties of weighted Besov spaces.
Proposition 1.4. For fixed $0 < p, q \leq \infty$,
\[ S(\mathbb{R}^n) = \bigcap_{\alpha, s \in \mathbb{R}} B^s_{pq}(\mathbb{R}^n, \langle x \rangle^\alpha) \] (4)
and
\[ S'(\mathbb{R}^n) = \bigcup_{\alpha, s \in \mathbb{R}} B^s_{pq}(\mathbb{R}^n, \langle x \rangle^\alpha). \]

Proof. Step 1. The inclusion
\[ S(\mathbb{R}^n) \subset \bigcap_{\alpha, s \in \mathbb{R}} B^s_{pq}(\mathbb{R}^n, \langle x \rangle^\alpha) \]
is clear.

To prove that any $f \in \bigcap_{\alpha, s \in \mathbb{R}} B^s_{pq}(\mathbb{R}^n, \langle x \rangle^\alpha)$ belongs to $S(\mathbb{R}^n)$, it is sufficient to show that for any fixed $N \in \mathbb{N}$ there are $\alpha(N) \in \mathbb{R}$ and $s(N) \in \mathbb{R}$ such that
\[ \sup_{|\beta| \leq N} \sup_{x \in \mathbb{R}^n} |D^{\beta} f(x)| \leq c \|f\|_{B^s_{pq}(\mathbb{R}^n, \langle x \rangle^\alpha)}. \]

For any multiindex $\beta$ there are polynomials $P_{\gamma}^{\beta}$, $\deg P_{\gamma}^{\beta} \leq 2N$ such that
\[ \langle x \rangle^{2N} D^{\beta} f(x) = \sum_{\gamma \leq \beta} D^{\gamma} [(P_{\gamma}^{\beta} f)(x)]. \]

Hence
\[ \sup_{|\beta| \leq N} \sup_{x \in \mathbb{R}^n} |\langle x \rangle^{2N} D^{\beta} f(x)| \leq \sup_{|\beta| \leq N} \sup_{|\gamma| \leq N} |D^{\gamma} [(P_{\gamma}^{\beta} f)(x)]| \]
\[ \leq \sup_{|\beta| \leq N} \sum_{|\gamma| \leq N} |D^{\gamma} [(P_{\gamma}^{\beta} f)(x)]| \]
\[ \leq \sup_{|\beta| \leq N} \sum_{|\gamma| \leq N} \|P_{\gamma}^{\beta} f\|_{C^N(\mathbb{R}^n)}. \] (5)

Due to the embedding theorems [3, ch. 2.7.1],
\[ \|P_{\gamma}^{\beta} f\|_{C^N(\mathbb{R}^n)} \leq c \|P_{\gamma}^{\beta} f\|_{B^{N + \frac{n}{q} + \varepsilon}_{pq}(\mathbb{R}^n)} \]
\[ = c \|P_{\gamma}^{\beta} (\langle x \rangle^{2N} f)\|_{B^{N + \frac{n}{q} + \varepsilon}_{pq}(\mathbb{R}^n)} \]
\[ \leq c \|C^{N + \frac{n}{q} + \varepsilon}(\mathbb{R}^n)\| \cdot \|\langle x \rangle^{2N} f\|_{B^{N + \frac{n}{q} + \varepsilon}_{pq}(\mathbb{R}^n)}. \] (6)

for any $\varepsilon > 0$. $\frac{P_{\gamma}^{\beta}}{(\langle x \rangle^{2N})}$ is a pointwise multiplier for $B^{N + \frac{n}{q} + \varepsilon}_{pq}(\mathbb{R}^n)$ [3, ch. 2.8.2]. Therefore
\[ \left\| \frac{P_{\gamma}^{\beta}}{(\langle x \rangle^{2N})} (\langle x \rangle^{2N} f) \right\|_{B^{N + \frac{n}{q} + \varepsilon}_{pq}(\mathbb{R}^n)} \]
\[ \leq c \left\| \frac{P_{\gamma}^{\beta}}{(\langle x \rangle^{2N})} \right\|_{C^{N + \frac{n}{q} + \varepsilon}(\mathbb{R}^n)} \cdot \left\| (\langle x \rangle^{2N} f) \right\|_{B^{N + \frac{n}{q} + \varepsilon}_{pq}(\mathbb{R}^n)}. \] (7)
According to Remark 1.3
\[
\left\| \langle x \rangle^{2N} f \right\|_{B_{pq}^{N+\frac{n}{p}+\varepsilon}(\mathbb{R}^n)} \sim \left\| f \right\|_{B_{pq}^{N+\frac{n}{p}+\varepsilon}(\mathbb{R}^n, \langle x \rangle^{2N})}.
\]
Combining (5)–(8), one gets
\[
\sup_{|\beta| \leq N} \sup_{x \in \mathbb{R}^n} \left\| \langle x \rangle^{2N} D^\beta f(x) \right\| \leq c \sum_{|\gamma| \leq N} \left\| \langle x \rangle^{2N} f \right\|_{B_{pq}^{N+\frac{n}{p}+\varepsilon}(\mathbb{R}^n)} \leq c \left\| f \right\|_{B_{pq}^{N+\frac{n}{p}+\varepsilon}(\mathbb{R}^n, \langle x \rangle^{2N})}.
\]
and it follows (4).

Step 2. Let \(1 < p \leq \infty, 1 < q \leq \infty\) and let \(p'\) and \(q'\) be defined in the standard way by
\[
\frac{1}{p} + \frac{1}{p'} = 1, \quad \frac{1}{q} + \frac{1}{q'} = 1.
\]
The inclusion
\[
\bigcup_{\alpha, s \in \mathbb{R}} B_{pq}^s(\mathbb{R}^n, \langle x \rangle^\alpha) \subset S'(\mathbb{R}^n)
\]
is evident.
As far as the opposite inclusion is concerned, we recall that \(f \in S'(\mathbb{R}^n)\) if and only if there are \(l \in \mathbb{N}\) and \(m \in \mathbb{N}\) such that
\[
\left| f(\varphi) \right| \leq c \sup_{|\alpha| \leq m} \sup_{x \in \mathbb{R}^n} \langle x \rangle^{l} |D^\alpha \varphi(x)|,
\]
for all \(\varphi \in S(\mathbb{R}^n)\). By (9),
\[
\sup_{|\alpha| \leq m} \sup_{x \in \mathbb{R}^n} \langle x \rangle^{l} |D^\alpha \varphi(x)| \leq c \left\| \varphi \right\|_{B_{pq}^{m+\frac{n}{p}+\varepsilon}(\mathbb{R}^n, \langle x \rangle^l)}.
\]
According to our choice of \(p\) and \(q\), it follows that \(1 \leq p' < \infty\) and \(1 \leq q' < \infty\). Thus, by [3, ch. 2.11.2],
\[
f \in \left( B_{p'q'}^{m+\frac{n}{p}+\varepsilon}(\mathbb{R}^n, \langle x \rangle^l) \right)' = B_{pq}^{-(m+\frac{n}{p}+\varepsilon)}(\mathbb{R}^n, \langle x \rangle^{-l}).
\]
This means
\[
S'(\mathbb{R}^n) \subset \bigcup_{\alpha, s \in \mathbb{R}} B_{pq}^s(\mathbb{R}^n, \langle x \rangle^\alpha).
\]
Step 3. Let \(0 < p \leq 1, 1 < q \leq \infty\). By the arguments above, for \(f \in S'(\mathbb{R}^n)\) there are \(\alpha \in \mathbb{R}\) and \(s \in \mathbb{R}\) such that
\[
f \in B_{\infty q}^s(\mathbb{R}^n, \langle x \rangle^\alpha).
\]
We want to show that

\[ f \in B_{pq}^s(\mathbb{R}^n, \langle x \rangle^{\alpha-\gamma}), \quad \gamma > \frac{n}{p}. \]

Indeed,

\[
\|f | B_{pq}^s(\mathbb{R}^n, \langle x \rangle^{\alpha-\gamma})\| = \left( \sum_{j=0}^{\infty} 2^{jsq} \|\langle x \rangle^{\alpha-\gamma}(\varphi_j \hat{f})\|_{L^p(\mathbb{R}^n)}^q \right)^{\frac{1}{q}} \leq c \|f | B_{\infty q}^s(\mathbb{R}^n, \langle x \rangle^\alpha)\|.
\]

**Step 4.** When \(0 < q \leq 1\), first we may find \(\alpha \in \mathbb{R}\) and \(s \in \mathbb{R}\) such that

\[ f \in B_{pq}^s(\mathbb{R}^n, \langle x \rangle^{\alpha}), \]

\(q^* > 1\), and then use the fact that

\[ B_{pq}^s(\mathbb{R}^n, \langle x \rangle^{\alpha}) \subset B_{pq}^{s-\varepsilon}(\mathbb{R}^n, \langle x \rangle^{\alpha}), \quad \varepsilon > 0. \]

Next we recall some notation. A measure \(\mu\) is called \(\sigma\)-finite in \(\mathbb{R}^n\) if for any \(R > 0\),

\[ \mu(\{x : |x| < R\}) < \infty. \]

A measure \(\mu\) is a Radon measure if all Borel sets are \(\mu\) measurable and

1. \(\mu(K) < \infty\) for compact sets \(K \subset \mathbb{R}^n\),
2. \(\mu(V) = \sup\{\mu(K) : K \subset V \text{ is compact}\}\) for open sets \(V \subset \mathbb{R}^n\),
3. \(\mu(A) = \inf\{\mu(V) : A \subset V, \text{ V is open}\}\) for \(A \subset \mathbb{R}^n\).

Let \(\mu\) be a positive Radon measure in \(\mathbb{R}^n\). Let \(T_{\mu}\),

\[ T_{\mu} : \varphi \mapsto \int_{\mathbb{R}^n} \varphi(x) \mu(dx), \quad \varphi \in S(\mathbb{R}^n), \]

be the linear functional generated by \(\mu\).

**Definition 1.5.** A positive Radon measure \(\mu\) is said to be tempered if \(T_{\mu} \in S'(\mathbb{R}^n)\).

**Proposition 1.6.** Let \(\mu^1\) and \(\mu^2\) be two tempered Radon measures. Then

\[ T_{\mu^1} = T_{\mu^2} \text{ in } S'(\mathbb{R}^n) \quad \text{if, and only if}, \quad \mu^1 = \mu^2. \]

**Proof.** The Proposition is valid by the arguments in [5, p. 80].
This justifies the identification of \( \mu \) and correspondent tempered distribution \( T_\mu \) and we may write \( \mu \in S'(\mathbb{R}^n) \).

**Definition 1.7.** \( f \in S'(\mathbb{R}^n) \) is called a positive distribution if

\[
f(\varphi) \geq 0 \quad \text{for any } \varphi \in S(\mathbb{R}^n) \text{ with } \varphi \geq 0.
\]

If \( f \in L^1_{\text{loc}}(\mathbb{R}^n) \) then \( f \geq 0 \) means \( f(x) \geq 0 \) almost everywhere.

**Remark 1.8.** If \( f \) is a positive distribution, then \( f \in C_0(\mathbb{R}^n) \) and it follows from the Radon-Riesz theorem that there is a tempered Radon measure \( \mu \) such that

\[
f(\varphi) = \int_{\mathbb{R}^n} \varphi(x) \mu(dx)
\]

[2, pp. 61, 62, 71, 75].

**2. Main assertions**

Our next result refers to tempered measures.

**Theorem 2.1.**

(i) A Radon measure \( \mu \) in \( \mathbb{R}^n \) is tempered if, and only if, there is a real number \( \beta \) such that \( \langle x \rangle^\beta \mu \) is finite.

(ii) Let \( \mu \) be a tempered Radon measure in \( \mathbb{R}^n \). Let \( j \in \mathbb{N} \),

\[
A_j = \{ x : 2^{j-1} \leq |x| \leq 2^{j+1} \}, \quad A_0 = \{ x : |x| \leq 2 \}.
\]

Then for some \( c > 0, \alpha \geq 0 \),

\[
\mu(A_k) \leq c2^{\alpha k} \quad \text{for all } k \in \mathbb{N}_0.
\]

**Proof. Step 1.** First we prove part (ii). Suppose that the assertion does not hold. Then for \( c = 1 \) and \( l \in \mathbb{N} \) there is \( k_l \in \mathbb{N}_0 \) such that

\[
\mu(A_{k_l}) > 2^{k_l}.
\]

As soon as it is found one \( k_l \) with (10), it follows that there are infinitely many \( k_l^m, m \in \mathbb{N} \), that satisfy (10).

With \( j \in \mathbb{N} \),

\[
A_j^* = \{ x : 2^{j-2} \leq |x| \leq 2^{j+2} \}, \quad A_0^* = \{ x : |x| \leq 4 \}.
\]

For \( l = 1 \) take any of \( k_l^m \), let it be \( k_1 \). For \( l = 2 \) choose \( k_2 \gg k_1 \) in such a way that \( A_{k_1}^* \) and \( A_{k_2}^* \) have an empty intersection. For arbitrary \( l \in \mathbb{N} \) take

\[
k_l \gg k_{l-1} \quad \text{and} \quad A_{k_{l-1}}^* \cap A_{k_l}^* = \emptyset.
\]
Let $\phi_0$ be a $C^\infty$ function on $\mathbb{R}^n$ with
\[
\phi_0(x) = 1, \quad |x| \leq 2 \quad \text{and} \quad \phi_0(x) = 0, \quad |x| \geq 4.
\]
Let $k \in \mathbb{N}$ and
\[
\phi_k(x) = \phi_0(2^{-k}x) - \phi_0(2^{-k+3}x), \quad x \in \mathbb{R}^n.
\]
Then we have
\[
supp \phi_k \subset A_k^*\quad \text{and} \quad \phi_k(x) = 1, \quad x \in A_k.
\]
Let
\[
\phi(x) = \sum_{l=1}^{\infty} 2^{-lk_l} \phi_{k_l}(x).
\]
For any fixed $N \in \mathbb{N}_0$
\[
\sup_{|\alpha| \leq N} \sup_{x \in \mathbb{R}^n} (1 + |x|^2)^N |D^\alpha \phi(x)|
= \sup_{|\alpha| \leq N} \sup_{x \in \mathbb{R}^n} (1 + |x|^2)^N \left| D^\alpha \left( \sum_{l=1}^{\infty} 2^{-lk_l} \phi_{k_l}(x) \right) \right|
\leq \sup_{l \in \mathbb{N}} \sup_{|\alpha| \leq N} \sup_{x \in \mathbb{R}^n} 2^{-lk_l} 2^{-|\alpha|} 2^{2(k_l-1)} N (1 + |x|^2)^N |D^\alpha \phi_1(2^{-k_l+1}x)|.
\]
The last inequality holds, since the functions $\phi_{k_l}$ have disjoint supports. With the change of variables
\[
x' = 2^{-k_l+1}x
\]
one gets
\[
\sup_{|\alpha| \leq N} \sup_{x \in \mathbb{R}^n} (1 + |x|^2)^N |D^\alpha \phi(x)|
\leq \sup_{l \in \mathbb{N}} \sup_{|\alpha| \leq N} 2^{-lk_l} 2^{-|\alpha|} 2^{2(k_l-1)} N \sup_{x \in \mathbb{R}^n} (1 + |x|^2)^N |D^\alpha \phi_1(x)|
\leq c \sup_{l \in \mathbb{N}} \sup_{|\alpha| \leq N} 2^{-k_l(l+|\alpha|-2N)} \leq c \sup_{l \in \mathbb{N}} 2^{-k_l(l-2N)}.
\]
Since $N$ is fixed and $l$ is tending to infinity, $2^{-k_l(l-2N)}$ is bounded. Thus $\phi \in S(\mathbb{R}^n)$.

According to the definition of tempered Radon measures
\[
\int_{\mathbb{R}^n} \psi(x) \mu(dx) < +\infty
\]
for any $\psi \in S(\mathbb{R}^n)$, but
\[
\int_{\mathbb{R}^n} \varphi(x) \mu(dx) \geq \sum_{l=1}^{\infty} \int_{A_{l,i}} \varphi(x) \mu(dx) \geq \sum_{l=1}^{\infty} 2^{-l_k} a^{k_l} = +\infty.
\]
This means that our assertion (10) is false.

Step 2. We prove part (i). Since $\langle x \rangle^\beta \mu$ is finite, it is tempered. Then $\mu$ is also tempered. To prove the other direction we take $\beta = -(\alpha + 1)$. Then we get
\[
\langle x \rangle^\beta \mu(\mathbb{R}^n) = \int_{\mathbb{R}^n} \langle x \rangle^{-(\alpha+1)} \mu(dx) \leq \sum_{k=0}^{\infty} \int_{A_k} \langle x \rangle^{-(\alpha+1)} \mu(dx) \leq c \sum_{k=0}^{\infty} 2^{-k(\alpha+1)} 2^{k\alpha} < \infty.
\]

In order to characterize finite Radon measures we define the positive cone $B_{pq}^+(\mathbb{R}^n)$ as the collection of all positive $f \in B_{pq}^+(\mathbb{R}^n)$.

**Theorem 2.2.** Let $M(\mathbb{R}^n)$ be the collection of all finite Radon measures. Then
\[
M(\mathbb{R}^n) = \hat{B}^0_{1\infty}(\mathbb{R}^n)
\]
and
\[
\mu(\mathbb{R}^n) \sim ||\mu| B^0_{1\infty}(\mathbb{R}^n)||, \quad \mu \in M(\mathbb{R}^n).
\]

**Proof.** By the proof in [5, pp. 82, 83, Proposition 1.127],
\[
||\mu| B^0_{1\infty}(\mathbb{R}^n)|| \leq \mu(\mathbb{R}^n) \quad \text{if} \quad \mu \in M(\mathbb{R}^n).
\]

In order to prove the converse inequality, one use the characterisation of Besov spaces via local means. Let $k_0$ be a $C^\infty$ non-negative function with
\[
supp k_0 \subset \{ x : |x| \leq 1 \} \quad \text{and} \quad \overline{k_0(0)} \neq 0.
\]
If $f \in \hat{B}^0_{1\infty}(\mathbb{R}^n)$, then $f = \mu$ is a tempered measure. By [5, p. 10, Theorem 1.10],
\[
||\mu| B^0_{1\infty}(\mathbb{R}^n)|| \geq c||k_0(1,\mu)|L_1(\mathbb{R}^n)|| = e \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} k_0(x-y) d\mu(y) dx.
\]
Applying Fubini’s theorem, one gets
\[
||\mu| B^0_{1\infty}(\mathbb{R}^n)|| \geq c\mu(\mathbb{R}^n).
\]
Corollary 2.3. Let $f \in L_1(\mathbb{R}^n)$ and $f(x) \geq 0$ almost everywhere. Then

$$\|f \|_{L_1(\mathbb{R}^n)} \sim \|f \|_{B^0_{1,\infty}(\mathbb{R}^n)}.$$ 

Proof. Let $\mu = f \mu_L$, where $\mu_L$ is the Lebesgue measure. Then

$$\mu(\mathbb{R}^n) = \int_{\mathbb{R}^n} f(x) \mu_L(dx) = \|f \|_{L_1(\mathbb{R}^n)}$$

and

$$\|\mu \|_{B^0_{1,\infty}(\mathbb{R}^n)} = \|f \|_{B^0_{1,\infty}(\mathbb{R}^n)}.$$ 

From (11) follows the statement in the Corollary. 

Proposition 2.4. There are functions $f_j \in L_1(\mathbb{R}^n)$ with

$$\text{supp} \ f_j \subset \{ y : |y| \leq 1 \}, \quad j \in \mathbb{N},$$

such that $\{f_j\}$ is a bounded set in $B^0_{1,\infty}(\mathbb{R}^n)$, but

$$\|f_j \|_{L_1(\mathbb{R}^n)} \to \infty \quad \text{if} \quad j \to \infty.$$

Proof. We may assume $n = 1$.

Let $a \in C^1(\mathbb{R})$ be an odd function with

$$\text{supp} \ a \subset \{ x : |x| \leq 2 \}, \quad a(x) \geq 0, \quad x \geq 0$$

and

$$\max_{-2 \leq x \leq 2} |a(x)| = |a(-1)| = a(1) = 1.$$ 

If $c = \max_{-2 \leq x \leq 2} |a'(x)|$, then $c \geq 1$. Define $a_0 \in C^1(\mathbb{R})$ by

$$a_0(x) = c^{-1} a(x).$$

Then one has for any $x \in \mathbb{R}$,

$$|a_0(x)| \leq c^{-1} \leq 1, \quad |a_0'(x)| \leq 1, \quad \text{and} \quad \int_{\mathbb{R}} a_0(x) \ dx = 0.$$
Define a function $a_\nu$, $\nu \in \mathbb{N}$, by

$$a_\nu(x) = 2^\nu a_0(2^\nu x).$$

Then

$$\text{supp } a_\nu \subset [-2^{-\nu+1}, 2^{-\nu+1}]$$

and

$$|a_\nu(x)| \leq c^{-1}2^\nu, \quad |a'_\nu(x)| \leq 2^{2\nu}, \quad \int_{\mathbb{R}} a_\nu(x) \, dx = 0.$$

According to [5, p. 12, Definition 1.15], $a_0$ is an $1_1$-atom and $a_\nu$ are $(0, 1)_{1,1}$-atoms. It follows from [4, Theorem 13.8] that $\sum_{\nu=1}^{\infty} a_\nu(x)$ converges in $S'(\mathbb{R}^n)$ and represents an element of $B_{1,\infty}^0(\mathbb{R}^n)$. Let $f = \sum_{\nu=1}^{\infty} a_\nu.$

Let

$$f_j(x) = \sum_{\nu=1}^{j} a_\nu(x).$$

Then $\text{supp } f_j \subset [-1, 1],

$$\|f_j \|_{L_1(\mathbb{R}^n)} \geq \int_{\mathbb{R}^n} f_j(x) \, dx = \int_{\mathbb{R}^n} \sum_{\nu=1}^{j} a_\nu(x) \, dx = j \int_{\mathbb{R}^n} a_0(x) \, dx \to \infty, \quad j \to \infty.$$}

On the other hand one has by the above atomic argument

$$\|f_j \|_{B_{1,\infty}^0(\mathbb{R}^n)} \leq 1 \quad \text{for } j \in \mathbb{N}. \quad \Box$$

**Corollary 2.5.** Not any characteristic function of a measurable subset of $\mathbb{R}^n$ is a pointwise multiplier in $B_{1,\infty}^0(\mathbb{R}^n)$.

**Proof.** Let $f \in L_1(\mathbb{R}^n)$ real. Let $M_+$ be a set of points $x$ such that $f(x) \geq 0$ and $M_- = \{ x : f(x) < 0 \}$. Then

$$\|f \|_{L_1(\mathbb{R}^n)} = \|\chi_{M_+} f \|_{L_1(\mathbb{R}^n)} + \|\chi_{M_-} f \|_{L_1(\mathbb{R}^n)},$$

where $\chi_{M_+}, \chi_{M_-}$ are characteristic functions of sets $M_+$ and $M_-$ respectively. One may apply Corollary 2.3 to the functions $\chi_{M_+} f$ and $\chi_{M_-} f$ and get

$$\|f \|_{L_1(\mathbb{R}^n)} \leq c\|\chi_{M_+} f \|_{B_{1,\infty}^0(\mathbb{R}^n)} + c\|\chi_{M_-} f \|_{B_{1,\infty}^0(\mathbb{R}^n)}.$$
If any characteristic function of a set in $\mathbb{R}^n$ would be a pointwise multiplier in $B^0_{1,\infty}(\mathbb{R}^n)$, then
\[
\|\chi_{M_+ f} | B^0_{1,\infty}(\mathbb{R}^n)\| \leq c\|f | B^0_{1,\infty}(\mathbb{R}^n)\|, \quad \|\chi_{M_- f} | B^0_{1,\infty}(\mathbb{R}^n)\| \leq c\|f | B^0_{1,\infty}(\mathbb{R}^n)\|,
\]
hence
\[
\|f | L_1(\mathbb{R}^n)\| \leq c\|f | B^0_{1,\infty}(\mathbb{R}^n)\|.
\]
Since for any function $f \in L_1(\mathbb{R}^n)$ holds
\[
\|f | B^0_{1,\infty}(\mathbb{R}^n)\| \leq c\|f | L_1(\mathbb{R}^n)\|,
\]
one gets
\[
\|f | L_1(\mathbb{R}^n)\| \sim \|f | B^0_{1,\infty}(\mathbb{R}^n)\|, \quad \text{for real } f \in L_1(\mathbb{R}^n).
\]
This can be also extended to complex functions $f \in L_1(\mathbb{R}^n)$. But according to the Proposition 2.4 this is not true.

\section*{References}


