Divergent Cesàro Means of Jacobi-Sobolev Expansions

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ABSTRACT

Let \( \mu \) be the Jacobi measure supported on the interval \([-1, 1]\). Let introduce the Sobolev-type inner product
\[
\langle f, g \rangle = \int_{-1}^{1} f(x)g(x) \, d\mu(x) + Mf(1)g(1) + Nf'(1)g'(1),
\]
where \( M, N \geq 0 \). In this paper we prove that, for certain indices \( \delta \), there are functions whose Cesàro means of order \( \delta \) in the Fourier expansion in terms of the orthonormal polynomials associated with the above Sobolev inner product are divergent almost everywhere on \([-1, 1]\).

Key words: Jacobi-Sobolev type polynomials, Fourier expansion, Cesàro mean.

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Introduction

Let \( d\mu(x) = (1 - x)^\alpha(1 + x)^\beta \, dx, \alpha > -1, \beta > -1, \) be the Jacobi measure supported on the interval \([-1, 1]\). Let \( f \) and \( g \) functions in \( L^2(\mu) \) such that there exists the first derivative in 1. We can introduce the discrete Sobolev-type inner product
\[
\langle f, g \rangle = \int_{-1}^{1} f(x)g(x) \, d\mu(x) + Mf(1)g(1) + Nf'(1)g'(1) \tag{1}
\]
where $M \geq 0$, $N \geq 0$. We denote by $\{q_{n}^{(\alpha, \beta)}\}_{n \geq 0}$ the sequence of orthonormal polynomials with respect to the inner product (1) (see [1]). These polynomials are known in the literature as Jacobi-Sobolev type polynomials. For $M = N = 0$, the classical Jacobi orthonormal polynomials appear. We will denote them $\{p_{n}^{(\alpha, \beta)}\}_{n \geq 0}$.

For every function $f$ such that $\langle f, q_{n}^{(\alpha, \beta)} \rangle$ exists for $n = 0, 1, \ldots$, the Fourier expansion in Jacobi-Sobolev type polynomials is

$$\sum_{n=0}^{\infty} c_{n}(f) q_{n}^{(\alpha, \beta)}(x),$$

where

$$c_{n}(f) = \langle f, q_{n}^{(\alpha, \beta)} \rangle.$$

The Cesàro means of order $\delta$ of the Fourier expansion (2) are defined by (see [9, p. 76–77])

$$\sigma_{\delta}^{N} f(x) = \sum_{n=0}^{N} A_{N-n}^{\delta} c_{n}(f) q_{n}^{(\alpha, \beta)}(x),$$

where $A_{k}^{\delta} = \binom{k+\delta}{k}$.

In this contribution we will prove that there are functions such that their Cesàro means of order $\delta$ diverge almost everywhere on $[-1, 1]$. A similar result, when $M = N = 0$, has been obtained in [6].

Notice that, for an appropriate function $f$, the study of the convergence of Fourier series in terms of the polynomials associated to the Sobolev inner product

$$\langle f, g \rangle = \int_{-1}^{1} f(x) g(x) \, d\mu(x) + M f(c) g(c) + N f'(c) g'(c)$$

when $c \in [-1, 1]$ has been presented [7] and when $c \in (1, \infty)$ in ([3, 4]) some analogous results have been deduced.

Throughout this paper positive constants are denoted by $c, c_{1}, \ldots$, and they may vary at every occurrence. The notation $u_{n} \sim v_{n}$ means $c_{1} \leq u_{n}/v_{n} \leq c_{2}$ for sufficiently large $n$, and by $u_{n} \cong v_{n}$ we mean that the sequence $u_{n}/v_{n}$ converges to 1.

1. Jacobi-Sobolev type polynomials

Some basic properties of the polynomials $q_{n}^{(\alpha, \beta)}$ (see [1]) that we will need in the sequel, are given in below:

$$q_{n}^{(\alpha, \beta)}(x) = A_{n} p_{n}^{(\alpha, \beta)}(x) + B_{n}(x-1) p_{n-1}^{(\alpha+2, \beta)}(x) + C_{n}(x-1)^{2} p_{n-2}^{(\alpha+4, \beta)}(x)$$

where
(i) if \( M > 0 \) and \( N > 0 \) then

\[
A_n \cong -cn^{-2\alpha-2}, \quad B_n \cong cn^{-2\alpha-2}, \quad C_n \cong 1,
\]

(ii) if \( M = 0 \) and \( N > 0 \) then

\[
A_n \cong -\frac{1}{\alpha+2}, \quad B_n \cong 1, \quad C_n \cong \frac{1}{\alpha+2}.
\]

(iii) if \( M > 0 \) and \( N = 0 \) then

\[
A_n \cong cn^{-2\alpha-2}, \quad B_n \cong 1, \quad C_n \cong 0.
\]

\[
|q_n^{(\alpha,\beta)}(1)| \sim \begin{cases} 
  n^{-\alpha-3/2} & \text{if } M > 0, \ N \geq 0, \\
  n^{\alpha+1/2} & \text{if } M = 0, \ N \geq 0.
\end{cases}
\]  

(4)

\[
(q_n^{(\alpha,\beta)})'(1) \sim n^{-\alpha-7/2} \quad \text{if } M \geq 0, \ N > 0.
\]  

(5)

\[
\max_{x \in [-1,1]} |q_n^{(\alpha,\beta)}(x)| \sim n^{\beta+1/2} \quad \text{if } -1/2 \leq \alpha \leq \beta.
\]  

(6)

\[
|q_n^{(\alpha,\beta)}(\cos \theta)| = \begin{cases} 
  O(\theta^{-\alpha-1/2}(\pi - \theta)^{-\beta-1/2}) & \text{if } c/n \leq \theta \leq \pi - c/n, \\
  O(n^{\alpha+1/2}) & \text{if } 0 \leq \theta \leq c/n, \\
  O(n^{\beta+1/2}) & \text{if } \pi - c/n \leq \theta \leq \pi,
\end{cases}
\]  

(7)

for \( \alpha \geq -1/2, \ \beta \geq -1/2, \ \text{and} \ n \geq 1. \)

The asymptotic behavior of \( q_n^{(\alpha,\beta)} \), when \( x \in [-1+\epsilon, 1-\epsilon] \) and \( \epsilon > 0 \), is given by

\[
q_n^{(\alpha,\beta)}(x) = s_n^{(\alpha,\beta)}(1-x)^{-\alpha/2-1/4}(1+x)^{-\beta/2-1/4} \cos(k\theta + \gamma) + O(n^{-1}),
\]  

(8)

where \( x = \cos \theta, \ k = n + \frac{\alpha+\beta+1}{2}, \ \gamma = -(\alpha+1)\frac{\pi}{4}, \ \text{and} \ \lim_{n \to \infty} s_n^{(\alpha,\beta)} = (\frac{2}{\pi})^{1/2}. \)

The Mehler-Heine formula for Jacobi orthonormal polynomials is (see [8, Theorem 8.1.1 and (4.3.4)])

\[
\lim_{n \to \infty} (-1)^n n^{-\beta-1/2} p_n^{(\alpha,\beta)} \left( \cos \left( \frac{\pi - \frac{z}{n}}{n} \right) \right) = 2^{-\frac{\alpha+\beta}{2}} (z/2)^{-\beta} J_{\beta}(z),
\]  

(9)

where \( \alpha, \ \beta \) are real numbers and \( J_{\beta}(z) \) is the Bessel function. This formula holds uniformly for \( |z| \leq R, \) for \( R \) a given positive real number. From (9)

\[
\lim_{n \to \infty} (-1)^n n^{-\beta-1/2} p_n^{(\alpha,\beta)} \left( \cos \left( \frac{\pi - \frac{z}{n}}{n + j} \right) \right) = 2^{-\frac{\alpha+\beta}{2}} (z/2)^{-\beta} J_{\beta}(z)
\]  

(10)

holds uniformly for \( |z| \leq R, \) \( R \) a fixed positive real number, and uniformly on \( j \in N \cup \{0\} \).
Lemma 1.1. Let $\alpha, \beta > -1$ and $M, N \geq 0$. There exists a positive constant $c$ such that
\[
\lim_{n \to \infty} (-1)^n n^{-\beta-1/2} q_n^{(\alpha, \beta)} \left( \cos \left( \frac{\pi - z}{n} \right) \right) = c(z/2)^{-\beta} J_\beta(z),
\]
uniformly for $|z| \leq R$, $R > 0$ fixed.

Proof. Here we will only analyze the case when $M = 0$ and $N > 0$. The proof of the other cases can be done in a similar way. From (3) we have
\[
(-1)^n n^{-\beta-1/2} q_n^{(\alpha, \beta)} \left( \cos \left( \frac{\pi - z}{n + j} \right) \right) = A_n (-1)^n n^{-\beta-1/2} p_n^{(\alpha, \beta)} \left( \cos \left( \frac{\pi - z}{n + j} \right) \right)
\]
where $j \in \mathbb{N} \cup \{0\}$.

Finally, if $n \to \infty$ and using (3) and (10) we get
\[
\lim_{n \to \infty} (-1)^n n^{-\beta-1/2} q_n^{(\alpha, \beta)} \left( \cos \left( \frac{\pi - z}{n + j} \right) \right) = \left( -\frac{1}{\alpha + 2} \right) 2^{\alpha+\beta} + 2 \cdot 2^{-\alpha+\beta+2} + \frac{1}{\alpha + 2} \cdot 4 \cdot 2^{-\beta-\alpha+4} \left( \frac{z}{2} \right)^{-\beta} J_\beta(z).
\]

For every function $f$ such that $\langle f, q_n^{(\alpha, \beta)} \rangle$ exists for $n = 0, 1, \ldots$, the Fourier-Sobolev coefficients of the series (2) can be written as
\[
c_n(f) = c_n'(f) + M f(1) q_n^{(\alpha, \beta)}(1) + N f'(1) (q_n^{(\alpha, \beta)})'(1),
\]
where
\[
c_n'(f) = \int_{-1}^{1} f(x) q_n^{(\alpha, \beta)}(x)(1 - x)^\alpha (1 + x)^\beta \, dx.
\]

Next, we will estimate the following integral involving Jacobi-Sobolev type polynomials
\[
\int_{-1}^{1} |q_n^{(\alpha, \beta)}(x)|^q (1 - x)^\alpha (1 + x)^\beta \, dx
\]
where $1 \leq q < \infty$. For $M = N = 0$ the calculation of this integral appears in [8, p. 391, Exercise 91] (see also [5, (2.2)]).

First we compute an upper bound for this integral:
Theorem 1.2. Let $M \geq 0$ and $N \geq 0$. For $\alpha \geq -1/2$

$$\int_0^1 (1 - x)^\alpha |q_n^{(\alpha, \beta)}(x)|^q \, dx = \begin{cases} O(1) & \text{if } 2\alpha > q\alpha - 2 + q/2, \\ O(\log n) & \text{if } 2\alpha = q\alpha - 2 + q/2, \\ O(n^{q\alpha + q/2 - 2\alpha - 2}) & \text{if } 2\alpha < q\alpha - 2 + q/2. \end{cases}$$

For $\beta \geq -1/2$

$$\int_{-1}^0 (1 + x)^\beta |q_n^{(\alpha, \beta)}(x)|^q \, dx = \begin{cases} O(1) & \text{if } 2\beta > q\beta - 2 + q/2, \\ O(\log n) & \text{if } 2\beta = q\beta - 2 + q/2, \\ O(n^{q\beta + q/2 - 2\beta - 2}) & \text{if } 2\beta < q\beta - 2 + q/2. \end{cases}$$

Proof. From (7), for $q\alpha + q/2 - 2\alpha - 2 \neq 0$, we have

$$\int_0^1 (1 - x)^\alpha |q_n^{(\alpha, \beta)}(x)|^q \, dx = O(1) \int_0^{\pi/2} \theta^{2\alpha + 1} |q_n^{(\alpha, \beta)}(\cos \theta)|^q \, d\theta$$

$$= O(1) \int_0^{\pi/2} \theta^{2\alpha + 1} n^{q\alpha + q/2} \, d\theta$$

$$+ O(1) \int_{n-1}^{\pi/2} \theta^{2\alpha + 1} \theta^{-q\alpha - q/2} \, d\theta$$

$$= O(n^{q\alpha + q/2 - 2\alpha - 2}) + O(1),$$

and for $q\alpha + q/2 - 2\alpha - 2 = 0$ we have

$$\int_0^1 (1 - x)^\alpha |q_n^{(\alpha, \beta)}(x)|^q \, dx = O(\log n).$$

For the proof of the second part we can proceed in a similar way. \qed

Now, a technique similar to the used in [8, Theorem 7.34] yields:

Theorem 1.3. Let $M \geq 0$ and $N \geq 0$. For $\beta > -1/2$

$$\int_{-1}^0 (1 + x)^\beta |q_n^{(\alpha, \beta)}(x)|^q \, dx \sim n^{q\beta + q/2 - 2\beta - 2}$$

where $\frac{4(\beta+1)}{2\beta+1} < q < \infty$.

Proof. For the proof of this theorem it is enough to find a lower bound for the integral.
Let $\beta \geq -1/2$, $M \geq 0$ and $N \geq 0$. According to Lemma 1.1, we have
\[
\int_{\pi/2}^{\pi} (\pi - \theta)^{2\beta + 1} |q_n^{(\alpha, \beta)}(\cos \theta)|^q \, d\theta > \int_{\pi/2}^{\pi} (\pi - \theta)^{2\beta + 1} |q_n^{(\alpha, \beta)}(\cos \theta)|^q \, d\theta
\]
\[
= \int_0^1 (z/n)^{2\beta + 1} |q_n^{(\alpha, \beta)}(\cos(z/n))|^q \, n^{-1} \, dz
\]
\[
\cong c \int_0^1 (z/n)^{2\beta + 1} n^{q\beta + q/2} [z/2]^{-\beta} J_\beta(z)^q \, n^{-1} \, dz
\]
\[
\sim n^{q\beta + q/2 - 2\beta - 2}.
\]

2. Divergent Cesàro means of Jacobi-Sobolev expansions

If the expansion (2) is Cesàro summable of order $\delta$ on a set, say $E$, of positive measure in $[-1, 1]$, then from [9, Theorem 3.1.22] (see also [6, Lemma 1.1]) we get
\[
|c_n(f) q_n^{(\alpha, \beta)}(x)| = O(n^{\delta}), \quad x \in E.
\]

From the Egorov’s theorem there exists a subset $E_1 \subset E$ of positive measure such that
\[
|c_n(f) q_n^{(\alpha, \beta)}(x)| = O(n^{\delta})
\]
uniformly for $x \in E_1$. Hence, from (8), we have
\[
|n^{-\delta} c_n(f) (\cos(k\theta + \gamma) + O(n^{-1}))| \leq c
\]
uniformly for $x = \cos \theta \in E_1$. Using the Cantor-Lebesgue Theorem, (see [6, subsection 1.5] as well as [9, p. 316]), we get
\[
\left|\frac{c_n(f)}{n^{\delta}}\right| \leq c, \quad \forall n \geq 1. \quad (12)
\]

Now we will prove our main result:

**Theorem 2.1.** Let $\alpha$, $\beta$, $p$, and $\delta$ be given numbers such that
\[
\beta > -1/2, \quad \frac{1}{2} \leq \alpha \leq \beta, \quad 1 \leq p < \frac{4(\beta + 1)}{2\beta + 3}, \quad 0 \leq \delta < \frac{2\beta + 2}{p} - \frac{2\beta + 3}{2}.
\]

There exists $f \in L^p(\mu)$, supported on $[-1, 0]$, whose Cesàro means $\sigma_N^\delta f(x)$ are divergent almost everywhere on $[-1, 1]$. 

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Proof. Assume that
\[ 1 \leq p < \frac{4(\beta + 1)}{2\beta + 3}, \quad \delta < \frac{2\beta + 2}{p} - \frac{2\beta + 3}{2}. \]

For \( q \) conjugate to \( p \), from the last inequalities, we get
\[ \frac{4(\beta + 1)}{2\beta + 1} < q \leq \infty, \quad \delta < \beta + \frac{1}{2} - \frac{2\beta}{q} - \frac{2}{q}. \]

For the linear functional \( c'_n(f) = \int_{-1}^{1} f(x)q_{n}^{(\alpha,\beta)}(x) \, d\mu(x) \), from the uniform boundedness principle, (6) and Theorem 1.3, it follows that there is \( f \in L^p(\mu) \), supported on \([-1,0]\), such that
\[ \frac{c'_n(f)}{n^\delta} \to \infty, \quad \text{when} \quad n \to \infty. \]

Hence, from (4), (5), and (11), we obtain
\[ \frac{c_n(f)}{n^\delta} \to \infty, \quad \text{when} \quad n \to \infty. \]

Since this result is contrary with (12), \( \sigma_N^\delta f(x) \) is divergent almost everywhere. \( \square \)

Remark 2.2. Using formulae in [2], which relate the Riesz and Cesàro means of order \( \delta \geq 0 \), we conclude that Theorem 2.1 also holds for Riesz means.

References