On Conjugacy of $p$-gonal Automorphisms of Riemann Surfaces

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ABSTRACT

The classical Castelnuovo-Severi theorem implies that for \( g > (p-1)^2 \), a \( p \)-gonal automorphism group of a cyclic \( p \)-gonal Riemann surface \( X \) of genus \( g \) is unique. Here we deal with the case \( g \leq (p-1)^2 \); we give a new and short proof of a result of González-Diez that a cyclic \( p \)-gonal Riemann surface of such genus has one conjugacy class of \( p \)-gonal automorphism groups in the group of automorphisms of \( X \).

Key words: automorphisms of Riemann surfaces, fixed points, ramified coverings of Riemann surfaces, hyperellipticity.

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Introduction

A compact Riemann surface \( X \) of genus \( g \geq 2 \) is said to be cyclic \( p \)-gonal if there is an automorphism \( \varphi \) of \( X \) of order \( p \) such that the orbit space \( X/\varphi \) is the Riemann sphere. Such automorphism is called \( p \)-gonal automorphism and it gives rise to a ramified covering of the Riemann sphere by \( X \) with \( p \) sheets. So Castelnuovo-Severi theorem [4] asserts that for \( g > (p-1)^2 \), the group generated by a \( p \)-gonal automorphism of a Riemann surface of genus \( g \) is unique as was mentioned by Accola in [1]. Here we

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prove that for \( g \leq (p - 1)^2 \), a cyclic \( p \)-gonal Riemann surface has one conjugacy class of \( p \)-gonal automorphism groups in the group \( \text{Aut}(X) \) of automorphisms of \( X \). This result has been proved using different techniques by González-Diez in [5].

We shall use combinatorial methods based on the Riemann uniformization theorem and combinatorial theory of Fuchsian groups as in [6] (see also [3]), where the reader can find necessary notions and facts.

1. On fixed points of automorphisms of Riemann surfaces

By the Riemann uniformization theorem an arbitrary compact Riemann surface of genus \( g \) can be represented as the orbit space \( \mathcal{H}/\Gamma \), where \( \mathcal{H} \) is the upper half plane and \( \Gamma \) is a Fuchsian surface group with signature \( (g; -) \). A group of automorphisms of a surface so given can be presented as \( \Lambda/\Gamma \) for some Fuchsian group \( \Lambda \). So the Riemann Hurwitz formula gives at once the following easy but useful result.

**Lemma 1.1.** A Riemann surface \( X = \mathcal{H}/\Gamma \) is cyclic \( p \)-gonal for a prime \( p \) if and only if there exists a Fuchsian group with signature \( (0; p, \ldots, p) \), where \( s = 2(g + p - 1)/(p - 1) \), containing \( \Gamma \) as a normal subgroup of index \( p \).

Observe that a \( p \)-gonal automorphism of a Riemann surface of genus \( g \) has \( 2(g + p - 1)/(p - 1) \) fixed points. We shall use the following theorem of Macbeath [7] concerning fixed points of automorphisms of Riemann surfaces.

**Theorem 1.2.** Let \( X = \mathcal{H}/\Gamma \) be a Riemann surface with the automorphism group \( G = \Lambda/\Gamma \) and let \( x_1, \ldots, x_r \) be a set of elliptic canonical generators of \( \Lambda \) whose periods are \( m_1, \ldots, m_r \) respectively. Let \( \theta : \Lambda \rightarrow G \) be the canonical epimorphism. Then the number \( F(\varphi) \) of points of \( X \) fixed by a nontrivial element \( \varphi \) of \( G \) is given by the formula

\[
F(\varphi) = |N_G(\langle \varphi \rangle)| \sum 1/m_i,
\]

where \( N \) stands for the normalizer and the sum is taken over those \( i \) for which \( \varphi \) is conjugate to a power of \( \theta(x_i) \).

Finally we shall use the following easy

**Lemma 1.3.** Let \( G \) be a finite group of order bigger than \( p^2 \) generated by two elements \( a, b \) of prime order \( p \). Then for the normalizer \( N \) of the group generated by \( a, b \), \( |N| \leq |G|/p \).

**Proof.** Clearly no nontrivial power of \( b \) belongs to \( N \) since otherwise \( |G| \leq p^2 \). So \( |G : N| \geq p \). \( \square \)
2. On \( p \)-gonal automorphisms of Riemann surfaces

As we mentioned before for \( g > (p - 1)^2 \), a \( p \)-gonal automorphism group of a cyclic \( p \)-gonal Riemann surface of genus \( g \) is unique. Here we deal with \( g \leq (p - 1)^2 \).

**Theorem 2.1.** A cyclic \( p \)-gonal Riemann surface of genus \( g \leq (p - 1)^2 \), has one conjugacy class of \( p \)-gonal automorphism groups in the group \( \text{Aut}(X) \) of automorphisms of \( X \).

**Proof.** Let \( X \) be a cyclic \( p \)-gonal Riemann surface of genus \( g \leq (p - 1)^2 \) and let \( \langle a_1 \rangle, \ldots, \langle a_m \rangle \) be representatives of all conjugacy classes of \( p \)-gonal automorphism groups. By the Riemann uniformization theorem \( X = \mathcal{H}/\Gamma \) and by a Sylow theorem \( a_1, \ldots, a_m \) can be assumed to belong to a \( p \)-subgroup of \( \text{Aut}(X) \). Assume, to get a contradiction, that \( m \geq 2 \), denote \( a_1 = a, a_2 = b \), and let \( G = \langle a, b \rangle \). Then \( G = \Lambda/\Gamma \), where \( \Lambda \) is a Fuchsian group with signature \( (h; m_1, \ldots, m_r) \). Let \( \theta \) be the canonical projection of \( \Lambda \to G \).

We shall show first that \( G \) has order \( p^2 \). In contrary assume that \( |G| = n > p^2 \). Then, by Lemma 1.3 and Theorem 1.2, every period of \( \Lambda \) produces at most \( n/p^2 \) fixed points of \( a \) or \( b \) and therefore in particular

\[
4(g + p - 1)/(p - 1) \leq rn/p^2. \tag{1}
\]

Now for \( h \neq 0 \), the area \( \mu(\Lambda) \) of \( \Lambda \) satisfies \( \mu(\Lambda) \geq 2\pi r(p - 1)/p \) and so, by (1) and the Hurwitz-Riemann formula, \( 2g - 2 \geq 4p(g + p - 1) \geq 12(g + 2) \). Thus \( h = 0 \). But then \( r \geq 3 \).

First, let \( r \geq 4 \). Then \( \mu(\Lambda) \geq 2\pi(-2 + r(p - 1)/p) \) and so, by the Hurwitz-Riemann formula, \( g \geq nr(p - 1)/2p - n + 1 \geq (n - 2)/p + 1 \geq n/3 + 1 \). On the other hand the Hurwitz-Riemann formula and (1) gives also \( 2g - 2 \geq -2n + 4p(g + p - 1) \geq -2n + 12g \) and so \( g < n/5 \), a contradiction.

Now let \( r = 3 \). Since \( a \) and \( b \) can not be simultaneously conjugate to a power of some \( \theta(x) \), only one proper period produces fixed points in \( a \) or in \( b \) by Theorem 1.2; assume that this is the case for \( a \). Then since \( a \) and \( b \) have the same number of fixed points, the remaining two proper periods may produce at most \( n/p^2 \) fixed points in \( b \). So (1) actually becomes

\[
4(g + p - 1)/(p - 1) \leq 2n/p^2. \tag{2}
\]

But for \( p \geq 5 \), \( \mu(\Lambda) \geq 2\pi(-2 + 3(p - 1)/p) \). Thus

\[
4\pi(g - 1) = n\mu(\Lambda) \\
\geq 2\pi(-2n + 3n(p - 1)/p) \\
\geq 2\pi(-2n + 6p(g + p - 1))
\]

and so \( g \leq n/14 \). On the other hand, by the Hurwitz-Riemann formula

\[
n = 4\pi(g - 1)/\mu(\Lambda) \leq 2p(g - 1)/(p - 3)
\]

which gives \( g \geq n/5 \), a contradiction.
For \( p = 3 \), a period of \( \Lambda \) is at least 9 since \((0; 3, 3, 3)\) is not a signature of a Fuchsian group. But then \( \mu(\Lambda) \geq 4\pi/9 \) and therefore by the Hurwitz-Riemann formula \( g \geq n/9 \), while by (2) \( g \leq n/9 - 2 \), a contradiction.

So we can assume that \( G \) has order \( p^2 \) and therefore \( G = \mathbb{Z}_p \oplus \mathbb{Z}_p \). Here \( \Lambda \) has signature \((h; p, \ldots, p)\), with \( h = 0 \) by the Hurwitz-Riemann formula. But then for \( r \geq 5 \), \( \mu(\Lambda) \geq 2\pi(3p - 5)/p \) and so the Hurwitz-Riemann formula gives \( g \geq p(3p - 5)/2 + 1 \) which is bigger than \((p - 1)^2\). So \( r \leq 4 \).

However, for \( r = 3 \), there is a Fuchsian group \( \Lambda' \) with signature \((0; 3, 3, p)\) containing \( \Lambda \) as a subgroup and \( \Gamma \) as a normal subgroup by [8] and by Theorem 5.2 (i) of [2] respectively. Furthermore by N6 of [2], all canonical generators of \( \Lambda \) are conjugate in \( \Lambda' \) and so all \( p \)-gonal automorphism groups of our surface are conjugate in \( \Lambda'/\Gamma \subseteq \text{Aut}(X) \), a contradiction.

The case \( r = 4 \) is similar. Here each \( \theta(x_i) \) is conjugate to a nontrivial power of \( a \) or \( b \) since otherwise \( a \) and \( b \) would have at most 3\( p \) fixed points in total, by Theorem 1.2, while on the other hand they should have 4\( p \) such points by Lemma 1.1, since by the Hurwitz-Riemann formula the genus of the corresponding surface equals \((p - 1)^2\). So, for some permutation \( \sigma \),

\[
\theta(x_{\sigma(1)}) = a^\alpha, \quad \theta(x_{\sigma(2)}) = a^{-\alpha}, \quad \theta(x_{\sigma(3)}) = b^\beta, \quad \theta(x_{\sigma(4)}) = b^{-\beta}.
\]

But then the mappings

\[
\theta(x_1) \mapsto \theta(x_2), \quad \theta(x_2) \mapsto \theta(x_1), \quad \theta(x_3) \mapsto \theta(x_4), \quad \theta(x_4) \mapsto \theta(x_3)
\]

and

\[
\theta(x_1) \mapsto \theta(x_4), \quad \theta(x_2) \mapsto \theta(x_3), \quad \theta(x_3) \mapsto \theta(x_2), \quad \theta(x_4) \mapsto \theta(x_1)
\]

induce automorphisms of \( G \). So by N4 of [2], we obtain that a nontrivial power of \( a \) is conjugated to a nontrivial power of \( b \) in \( \text{Aut}(X) \). This is a contradiction which completes the proof.

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References

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*p-gonal automorphisms of Riemann surfaces*


