A New Proof of the Jawerth-Franke Embedding

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ABSTRACT

We present an alternative proof of the Jawerth embedding

$$F_{p_0}^{s_0}(\mathbb{R}^n) \hookrightarrow B_{p_1}^{s_1}(\mathbb{R}^n),$$

where

$$-\infty < s_1 < s_0 < \infty, \quad 0 < p_0 < p_1 \leq \infty, \quad 0 < q \leq \infty$$

and

$$s_0 - \frac{n}{p_0} = s_1 - \frac{n}{p_1}.$$

The original proof given in [3] uses interpolation theory. Our proof relies on wavelet decompositions and transfers the problem from function spaces to sequence spaces. Using similar techniques, we also recover the embedding of Franke [2].

Key words: Besov spaces, Triebel-Lizorkin spaces, Sobolev embedding, Jawerth-Franke embedding.

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Introduction

Let $B_{p_0}^{s_0}(\mathbb{R}^n)$ and $F_{p_0}^{s_0}(\mathbb{R}^n)$ denote the Besov and Triebel-Lizorkin function spaces, respectively. The classical Sobolev embedding theorem can be extended to these two scales.

**Theorem 0.1.** Let $-\infty < s_1 < s_0 < \infty$ and $0 < p_0 < p_1 \leq \infty$ with
\[
s_0 = \frac{n}{p_0} = s_1 - \frac{n}{p_1}.
\]

(i) If $0 < q_0 \leq q_1 \leq \infty$, then
\[B_{p_0}^{s_0}(\mathbb{R}^n) \hookrightarrow B_{p_1}^{s_1}(\mathbb{R}^n).\]

(ii) If $0 < q_0, q_1 \leq \infty$ and $p_1 < \infty$, then
\[F_{p_0}^{s_0}(\mathbb{R}^n) \hookrightarrow F_{p_1}^{s_1}(\mathbb{R}^n).\]

We observe that there is no condition on the fine parameters $q_0, q_1$ in (2). This surprising effect was first observed in full generality by Jawerth, [3]. Using (2), we may prove
\[F_{p_0}^{s_0}(\mathbb{R}^n) \hookrightarrow F_{p_1}^{s_1}(\mathbb{R}^n) = B_{p_1}^{s_1}(\mathbb{R}^n)\]
and
\[B_{p_0}^{s_0}(\mathbb{R}^n) = F_{p_0}^{s_0}(\mathbb{R}^n) \hookrightarrow F_{p_1}^{s_1}(\mathbb{R}^n)\]
for every $0 < q \leq \infty$. But Jawerth [3] and Franke [2] showed that these embeddings are not optimal and may be improved.

**Theorem 0.2.** Let $-\infty < s_1 < s_0 < \infty$, $0 < p_0 < p_1 \leq \infty$, and $0 < q \leq \infty$ with (1).

(i) Then
\[F_{p_0}^{s_0}(\mathbb{R}^n) \hookrightarrow B_{p_1}^{s_1}(\mathbb{R}^n).\]

(ii) If $p_1 < \infty$, then
\[B_{p_0}^{s_0}(\mathbb{R}^n) \hookrightarrow F_{p_1}^{s_1}(\mathbb{R}^n).\]

The original proofs (see [2, 3]) use interpolation techniques. We rely on a different method. First, we observe that using (for example) the wavelet decomposition method, (3) and (4) are equivalent to
\[f_{p_0}^{s_0} \hookrightarrow b_{p_1}^{s_1} \quad \text{and} \quad b_{p_0}^{s_0} \hookrightarrow f_{p_1}^{s_1}\]
under the same restrictions on parameters $s_0$, $s_1$, $p_0$, $p_1$, $q$ as in Theorem 0.2. Here, $b_{pq}$ and $f_{pq}$ stands for the sequence spaces of Besov and Triebel-Lizorkin type. We prove (5) directly using the technique of non-increasing rearrangement on a rather elementary level.

All the unimportant constants are denoted by the letter $c$, whose meaning may differ from one occurrence to another. If $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are two sequences of positive real numbers, we write $a_n \lesssim b_n$ if, and only if, there is a positive real number $c > 0$ such that $a_n \leq cb_n$, $n \in \mathbb{N}$. Furthermore, $a_n \approx b_n$ means that $a_n \lesssim b_n$ and simultaneously $b_n \lesssim a_n$.

1. Notation and definitions

We introduce the sequence spaces associated with the Besov and Triebel-Lizorkin spaces. Let $m \in \mathbb{Z}^n$ and $\nu \in \mathbb{N}_0$. Then $Q_{\nu m}$ denotes the closed cube in $\mathbb{R}^n$ with sides parallel to the coordinate axes, centred at $2^{-\nu}m$, and with side length $2^{-\nu}$. By $\chi_{\nu m} = \chi_{Q_{\nu m}}$ we denote the characteristic function of $Q_{\nu m}$. If $\lambda = \{ \lambda_{\nu m} : \nu \in \mathbb{N}_0, m \in \mathbb{Z}^n \}$, $-\infty < s < \infty$, and $0 < p, q \leq \infty$, we set

$$\|\lambda\|_{b_{pq}} = \left( \sum_{\nu=0}^{\infty} 2^{\nu (s - \frac{n}{p}) q} \left( \sum_{m \in \mathbb{Z}^n} |\lambda_{\nu m}|^p \right)^{\frac{1}{p}} \right)^{\frac{1}{q}},$$

appropriately modified if $p = \infty$ and/or $q = \infty$. If $p < \infty$, we define also

$$\|\lambda\|_{f_{pq}} = \left\| \left( \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} 2^{\nu s} |\lambda_{\nu m}| \chi_{\nu m}(\cdot)^q \right)^{1/q} \right\|_{L_p(\mathbb{R}^n)}.$$

The connection between the function spaces $B_{pq}^s(\mathbb{R}^n)$, $F_{pq}^s(\mathbb{R}^n)$ and the sequence spaces $b_{pq}$, $f_{pq}$ may be given by various decomposition techniques, we refer to [7, chapters 2 and 3] for details and further references.

As a result of these characterizations, (3) is equivalent to (5).

We use the technique of non-increasing rearrangement. We refer to [1, chapter 2] for details.

**Definition 1.1.** Let $\mu$ be the Lebesgue measure in $\mathbb{R}^n$. If $h$ is a measurable function on $\mathbb{R}^n$, we define the non-increasing rearrangement of $h$ through

$$h^*(t) = \sup\{ \lambda > 0 : \mu\{ x \in \mathbb{R}^n : |h(x)| > \lambda \} > t \}, \quad t \in (0, \infty).$$

We denote its averages by

$$h^{**}(t) = \frac{1}{t} \int_0^t h^*(s) \, ds, \quad t > 0.$$
We shall use the following properties. The first two are very well known and their proofs may be found in [1, Proposition 1.8 in chapter 2, Theorem 3.10 in chapter 3].

**Lemma 1.2.** If $0 < p \leq \infty$, then

$$\|h \mid L_p(\mathbb{R}^n)\| = \|h^* \mid L_p(0, \infty)\|$$

for every measurable function $h$.

**Lemma 1.3.** If $1 < p \leq \infty$, then there is a constant $c_p$ such that

$$\|h^{**} \mid L_p(0, \infty)\| \leq c_p \|h^* \mid L_p(0, \infty)\|$$

for every measurable function $h$.

**Lemma 1.4.** Let $h_1$ and $h_2$ be two non-negative measurable functions on $\mathbb{R}^n$. If $1 \leq p \leq \infty$, then

$$\|h_1 + h_2 \mid L_p(\mathbb{R}^n)\| \leq \|h_1^* + h_2^* \mid L_p(0, \infty)\|.$$  

*Proof.* The proof follows from Theorems 3.4 and 4.6 in [1, chapter 2].

### 2. Main results

In this part, we present a direct proof of the discrete versions of Jawerth and Franke embedding. We start with the Jawerth embedding.

**Theorem 2.1.** Let $-\infty < s_1 < s_0 < \infty$, $0 < p_0 < p_1 \leq \infty$, and $0 < q \leq \infty$. Then

$$f_{p_0 q}^{s_0} \hookrightarrow b_{p_1 p_0}^{s_1} \text{ if } s_0 - \frac{n}{p_0} = s_1 - \frac{n}{p_1}.$$  

*Proof.* Using the elementary embedding

$$f_{p_0 q}^{s} \hookrightarrow f_{p q}^{*} \text{ if } 0 < q_0 \leq q_1 \leq \infty$$  

and the lifting property of Besov and Triebel-Lizorkin spaces (which is even simpler in the language of sequence spaces), we may restrict ourselves to the proof of

$$f_{p_0 \infty}^{s} \hookrightarrow b_{p_1 p_0}^{0}, \text{ where } s = n\left(\frac{1}{p_0} - \frac{1}{p_1}\right).$$

Let $\lambda \in f_{p_0 \infty}^{s}$ and set

$$h(x) = \sup_{\nu \in \mathbb{N}_0} 2^{ns} \sum_{m \in \mathbb{Z}^n} |\lambda_{\nu m}| \chi_{\nu m}(x).$$

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Hence

\[ |\lambda_{\nu m}| \leq 2^{-\nu s} \inf_{x \in Q_{\nu m}} h(x), \quad \nu \in \mathbb{N}_0, \quad m \in \mathbb{Z}^n. \]

Using this notation,

\[ \|\lambda| f^s_{p_0, \infty} \| = \|h| L_{p_0}(\mathbb{R}^n)\| \]

and

\[ \|\lambda| b^0_{p_1, p_0} \|_{p_0} \leq \sum_{\nu=0}^{\infty} 2^{-\nu n} \left( \sum_{m \in \mathbb{Z}^n} \inf_{x \in \mathbb{R}^n} h(x) \right)^{p_0/p_1} \]

\[ \leq \sum_{\nu=0}^{\infty} 2^{-\nu n} \left( \sum_{k=1}^{\infty} h^*(2^{-\nu n} k)^{p_1} \right)^{p_0/p_1}. \]

Using the monotonicity of \( h^* \) and \( p_0 < p_1 \) we get

\[ \|\lambda| b^0_{p_1, p_0} \|_{p_0} \leq \sum_{\nu=0}^{\infty} 2^{-\nu n} \left( \sum_{l=0}^{\nu} 2^n \cdot (2^n - 1) \cdot h^*(2^{-\nu n} 2^n l)^{p_0/p_1} \right)^{p_0/p_1} \]

\[ \approx \sum_{\nu=0}^{\infty} 2^{-\nu n} \left( \sum_{l=0}^{\nu} 2^n h^*(2^{-\nu n} 2^n l)^{p_0} \right)^{p_0/p_1}. \]

We substitute \( j = l - \nu \) and obtain

\[ \|\lambda| b^0_{p_1, p_0} \|_{p_0} \leq \sum_{j=-\infty}^{\nu} \sum_{\nu=-j}^{\infty} 2^{-n(\nu+j) p_0/p_1} h^*(2^n j)^{p_0} \]

\[ = \sum_{j=-\infty}^{\infty} 2^n h^*(2^n j)^{p_0} \sum_{\nu=-j}^{\infty} 2^{n\nu} \left( \frac{p_0}{p_1} - 1 \right) \]

\[ \approx \sum_{j=-\infty}^{\infty} 2^n h^*(2^n j)^{p_0} \approx \|h| L_{p_0}(0, \infty)\|_{p_0} = \|h| L_{p_0}(\mathbb{R}^n)\|_{p_0}. \]

If \( p_1 = \infty \), only notational changes are necessary. \( \Box \)

**Theorem 2.2.** Let \( -\infty < s_1 < s_0 < \infty, 0 < p_0 < p_1 < \infty, \) and \( 0 < q \leq \infty \). Then

\[ b^0_{p_0, p_1} \hookrightarrow f^s_{p_1, q} \quad \text{if} \quad \frac{n}{p_0} = \frac{s_1}{p_1}. \]

**Proof.** Using the lifting property and (6), we may suppose that \( s_1 = 0 \) and \( 0 < q < p_0 \).

By Lemma 1.4, we observe that

\[ \|\lambda| f^0_{p_1, q} \| = \left( \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} |\lambda_{\nu m}| \eta_{\nu m}(x) \right)^{1/q} \left\| L_{p_1}(\mathbb{R}^n) \right\| \]
may be estimated from above by
\[ \left\| \sum_{\nu=0}^{\infty} \sum_{m=0}^{\infty} \tilde{\lambda}_{\nu m} \tilde{\chi}_{\nu m}(\cdot) \right\|_{L_{\frac{q}{p}}(0, \infty)}^{1/q}, \]  \(7\)

where \(\tilde{\lambda}_{\nu} = \{\tilde{\lambda}_{\nu m}\}_{m=0}^{\infty}\) is a non-increasing rearrangement of \(\lambda_{\nu} = \{\lambda_{\nu m}\}_{m \in \mathbb{Z}^n}\) and \(\tilde{\chi}_{\nu m}\) is a characteristic function of the interval \((2^{-\nu} m, 2^{-\nu} m + 1)\).

Using duality, (7) may be rewritten as
\[ \sup_{g} \left( \int_{0}^{\infty} g(x) \left( \sum_{\nu=0}^{\infty} \sum_{m=0}^{\infty} \tilde{\lambda}_{\nu m} \tilde{\chi}_{\nu m}(x) \right) dx \right)^{1/q} = \sup_{g} \left( \sum_{\nu=0}^{\infty} \sum_{m=0}^{\infty} 2^{-\nu n} \tilde{\lambda}_{\nu m} \tilde{\chi}_{\nu m}(x) g_{\nu m} \right)^{1/q}, \]
\(8\)

where the supremum is taken over all non-increasing non-negative measurable functions \(g\) with \(\|g\|_{L_{\frac{q}{p}}(0, \infty)} \leq 1\) and \(g_{\nu m} = 2^{-\nu n} \int g(x) \tilde{\chi}_{\nu m}(x) \, dx\). Here, \(\beta\) is the conjugated index to \(p\). Similarly, \(\alpha\) stands for the conjugated index to \(\frac{q}{p}\).

We use twice Hölder’s inequality and estimate (8) from above by
\[ \left( \sum_{\nu=0}^{\infty} 2^{-\nu n} \left( \sum_{m=0}^{\infty} \tilde{\lambda}_{\nu m}^{p_1} \tilde{\chi}_{\nu m}(x) \right) \right)^{1/p_1} \cdot \sup_{g} \left( \sum_{\nu=0}^{\infty} \sum_{m=0}^{\infty} 2^{-\nu n} \tilde{\lambda}_{\nu m} \tilde{\chi}_{\nu m}(x) g_{\nu m} \right)^{1/q}, \]
\(9\)

Since \(s_0 = n \left( \frac{1}{p_0} - \frac{1}{p_1} \right) \) and \(p_1 (s_0 - \frac{s_0}{p_0}) = -n\), the first factor in (9) is equal to \(\|\lambda\|_{b^{s_0}_{p_0 p_1}}\). To finish the proof, we have to show that there is a number \(c > 0\) such that
\[ \left( \sum_{\nu=0}^{\infty} 2^{-\nu n} \left( \sum_{m=0}^{\infty} g_{\nu m}^{\alpha} \right)^{\frac{s_0}{p_0}} \right)^{\frac{p_1}{p_0}} \leq c \]
\(10\)

holds for every non-increasing non-negative measurable functions \(g\) with \(\|g\|_{L_{\beta}(0, \infty)} \leq 1\). We fix such a function \(g\). Using the monotonicity of \(g\), we get
\[ \sum_{m=0}^{\infty} g_{\nu m}^{\alpha} = \sum_{l=0}^{\infty} \sum_{m=2^{l n}-1}^{2^{l+1} n} \left( 2^{\nu n} \int_{2^{-\nu n} m}^{2^{-\nu n} (m+1)} g(x) \, dx \right)^{\alpha} \]
\[ \lesssim \sum_{l=0}^{\infty} 2^{ln} \left( \int_{2^{-\nu n} (2^{l n}-1)}^{2^{-\nu n} (2^{l+1} n)} g(x) \, dx \right)^{\alpha} \leq \sum_{l=0}^{\infty} 2^{ln} (g^{**})^{\alpha} (2^{l-\nu} n). \]
We use \(1 < \beta < \alpha\), Lemma 1.3 and obtain
\[
\left( \sum_{\nu=0}^{\infty} 2^{-\nu n} \left( \sum_{m=0}^{\infty} g_{\nu m}^\alpha \right)^\beta \right)^{1/\beta} \leq \left( \sum_{\nu=0}^{\infty} 2^{-\nu n} \left( \sum_{l=0}^{\infty} g^{\alpha \nu m}_{2^l-\nu} (2^l)^n \right) \right)^{1/\beta} \\
\leq \left( \sum_{\nu=0}^{\infty} 2^{-\nu n} \sum_{l=0}^{\infty} 2^{l n \frac{\beta}{\alpha}} (g^{**})^\beta (2^l)^n \right)^{1/\beta} \\
\leq \left( \sum_{k=-\infty}^{\infty} 2^{\beta n \frac{\beta}{\alpha}} \sum_{\nu=-k}^{\infty} 2^{\nu n (\frac{\beta}{\alpha}-1)} (g^{**})^\beta (2^k)^n \right)^{1/\beta} \\
\lesssim \left( \sum_{k=-\infty}^{\infty} 2^{\beta n} (g^{**})^\beta (2^k)^n \right)^{1/\beta} \\
\lesssim \|g^{**}| L_\beta (0, \infty)\| \leq c \|g | L_\beta (0, \infty)\| \leq c.
\]
Taking the \(\frac{1}{q}\)-power of this estimate, we finish the proof of (10).

The Theorems 2.1 and 2.2 are sharp in the following sense.

**Theorem 2.3.** Let \(-\infty < s_1 < s_0 < \infty\), \(0 < p_0 < p_1 \leq \infty\), and \(0 < q_0, q_1 \leq \infty\) with
\[
s_0 = \frac{n}{p_0 + s_1} = \frac{n}{p_1}.
\]
(i) If
\[
f^s_{p_0 q_0} \hookrightarrow b^s_{p_1 q_1}.
\]
then \(q_1 \geq p_0\).
(ii) If \(p_1 < \infty\) and
\[
b^{s_0}_{p_0 q_0} \hookrightarrow f^{s_1}_{p_1 q_1}.
\]
then \(q_0 \leq p_1\).

**Remark 2.4.** Using (any of) the usual decomposition techniques, the same statements hold true also for the function spaces. These results were first proved in [4].

**Proof.** (i) Suppose that \(0 < q_1 < p_0 < \infty\) and set
\[
\lambda_{\nu m} = \begin{cases} 
\nu^{-\frac{1}{\beta}} 2^{\nu t (\frac{\alpha}{\beta} - s_1)} & \text{if } \nu \in \mathbb{N} \text{ and } m = 0, \\
0, & \text{otherwise}.
\end{cases}
\]
A simple calculation shows that \(\|\lambda | f^s_{p_0 q_0}\| < \infty\) and \(\|\lambda | b^{s_1}_{p_1 q_1}\| = \infty\). Hence, (11) does not hold.
(ii) Suppose that \(0 < p_1 < q_0 \leq \infty\) and set
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\[ \lambda_{\nu m} = \begin{cases} \nu^{-\frac{1}{pt}} 2^{\nu(\frac{n}{pt} - s_1)} & \text{if } \nu \in \mathbb{N} \text{ and } m = 0, \\ 0, & \text{otherwise.} \end{cases} \]

Again, it is a matter of simple calculation to show, that \( \| \lambda \|_{b^{s_0}_{p_0,q_0}} < \infty \) and \( \| \lambda \|_{f^{s_1}_{p_1,q_1}} = \infty \). Hence, (12) is not true.

References


