Feller Semigroups Obtained by Variable Order Subordination

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ABSTRACT

For certain classes of negative definite symbols $q(x, \xi)$ and state space dependent Bernstein function $f(x, s)$ we prove that $-p(x, D)$, the pseudo-differential operator with symbol $-p(x, \xi) = -f(x, q(x, \xi))$, extends to the generator of a Feller semigroup. Our result extends previously known results related to operators of variable (fractional) order of differentiation, or variable order fractional powers. New concrete examples are given.

Key words: Feller semigroups, subordination in the sense of Bochner, pseudo-differential operators with negative definite symbols of variable order, Hoh’s symbolic calculus.

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Introduction

In the early days of the theory of pseudo-differential operators, pseudo differential operators of variable order had already been studied, compare A. Unterberger and J. Bokobza [21]. These considerations were taken up by H.-G. Leopold [16,17] who gave more emphasis on the function space point of view. On the other hand, also in the early days of the theory of pseudo-differential operators Ph. Courrège [2] pointed out that (most) generators of Feller semigroups are pseudo-differential operators, but

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their symbols do not belong to “nice” or “classical” symbol classes. Indeed, on \( S(\mathbb{R}^n) \) the generator of a Feller semigroup has the representation
\[
Au(x) = -q(x, D)u(x) = -(2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix \cdot \xi} q(x, \xi) \hat{u}(\xi) \, d\xi
\]
where the symbol \( q : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C} \) is measurable and locally bounded and for \( x \in \mathbb{R}^n \) fixed \( q(x, \cdot) \) is a continuous negative definite function, i.e., we have the Lévy-Khinchin representation
\[
q(x, \xi) = c(x) + id(x)\xi + \sum_{k,l=1}^{n} a_{k,l}(x)\xi_k \xi_l + \int_{\mathbb{R}^n \setminus \{0\}} \left( 1 - e^{-iy \cdot \xi} - \frac{iy \cdot \xi}{1 + |y|^2} \right) \nu(x, dy)
\]
with \( c(x) \geq 0, d(x) \in \mathbb{R}^n, a_{k,l}(x) \in \mathbb{R} \) and \( \sum_{k,l=1}^{n} a_{k,l}(x)\xi_k \xi_l \geq 0 \), and \( \int_{\mathbb{R}^n \setminus \{0\}} (1 + |y|^2) \nu(x, dy) < \infty \). Thus these symbols need not to be smooth with respect to \( \xi \) nor do they need to have a nice expansion into homogeneous functions. Maybe the fact that these symbols are a bit exotic is the reason why Courrèges result was almost ignored for around 25 years. In \cite{10}, see also \cite{9}, Courrèges idea was taken up and a systematic study of pseudo-differential operators generating Markov processes was initiated, see also \cite{11–13}.

The fact that the composition of a Bernstein function \( f \) with a continuous negative definite function \( \psi \) is again a continuous negative definite function gives a powerful tool to construct new (Feller) semigroups from given ones. If \( q(x, \xi) \) is a suitable symbol such that \(-q(x, D)\) generates a Feller semigroup, then \((f \circ q)(x, \xi) = f(q(x, \xi))\) is a symbol with the property that \( \xi \to (f \circ q)(x, \xi) \) is a continuous negative definite function and therefore \(-{(f \circ q)}(x, D)\) is a candidate for being a generator of a Feller semigroup. Of course, this procedure is closely linked to subordination in the sense of Bochner.

In a joint paper \cite{14} with H.-G. Leopold it was suggested to study Feller semigroups obtained by subordination of variable order, more precisely, to consider “fractional powers of variable order” in case of the symbol \((1 + |\xi|^2)^\alpha(x)\), i.e., to study \((x, \xi) \to (1 + |\xi|^2)^\alpha(x)\). These ideas were taken up and further investigations on fractional powers of variable order are due to A. Negoro \cite{20}, K. Kikuchi and A. Negoro \cite{15}, as well as F. Baldus \cite{1}. Finally, W. Hoh in \cite{7} could combine his symbolic calculus \cite{5} with these ideas, compare W. Hoh \cite{6, 8}.

The purpose of this note is twofold. First we suggest a method to study “variable order subordination” for more general Bernstein functions than \( f_\alpha(s) = s^\alpha \), \( 0 < \alpha < 1 \). More precisely, we consider symbols of the form
\[
p(x, \xi) = f(x, q(x, \xi))
\]
where \( q \) is a suitable symbol from Hoh’s class and \( f : \mathbb{R}^n \times [0, \infty) \to \mathbb{R} \) is a smooth function such that for fixed \( x \in \mathbb{R}^n \) the function \( s \to f(x, s) \) is a Bernstein function.
Our method uses some ideas from the theory of t-coercive (differential) operators as investigated by I. S. Louhivaara and C. Simader [18,19] in order to establish the result that \(-p(x,D)\) generates a Feller semigroup. Secondly, we enrich the class of examples by studying the Bernstein function

\[ s \rightarrow s^\frac{\alpha}{2} (1 - e^{-4s^{\frac{\alpha}{2}}}). \]

Since we depend on Hoh’s symbolic calculus we recollect some basic facts of this calculus in our first section. All our methods are standard, i.e., they are as in [11–13].

1. Hoh’s symbolic calculus

Before starting with our main considerations we need to recollect some basic results from Hoh’s symbolic calculus, see W. Hoh [5] or [6], compare also [12].

**Definition 1.1.** A continuous negative definite function \(\psi : \mathbb{R}^n \rightarrow \mathbb{R}\) belongs to the class \(\Lambda\) if for all \(\alpha \in \mathbb{N}_0^n\) it satisfies

\[ |\partial^\alpha (1 + \psi(\xi))| \leq c_{|\alpha|} (1 + \psi(\xi))^{2 - \rho(|\alpha|)}, \]

where \(\rho(k) = k \wedge 2\) for \(k \in \mathbb{N}_0^n\).

**Definition 1.2.**

(i) Let \(m \in \mathbb{R}\) and \(\psi \in \Lambda\). We then call a \(C^\infty\)-function \(q : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{C}\) a symbol in the class \(S^{m,\psi}_0(\mathbb{R}^n)\) if for all \(\alpha, \beta \in \mathbb{N}_0^n\) there are constants \(c_{\alpha,\beta} \geq 0\) such that

\[ |\partial^\beta \xi \partial^\alpha x q(x,\xi)| \leq c_{\alpha,\beta} (1 + \psi(\xi))^{m - \rho(|\alpha|)} \]

holds for all \(x \in \mathbb{R}^n\) and \(\xi \in \mathbb{R}^n\). We call \(m \in \mathbb{R}\) the order of the symbol \(q(x,\xi)\).

(ii) Let \(\psi \in \Lambda\) and suppose that for an arbitrarily often differentiable function \(q : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{C}\) the estimate

\[ |\partial^\beta \xi \partial^\alpha x q(x,\xi)| \leq c_{\alpha,\beta} (1 + \psi(\xi))^{\frac{m}{2}} \]

holds for all \(\alpha, \beta \in \mathbb{N}_0^n\) and \(x, \xi \in \mathbb{R}^n\). In this case we call \(q\) a symbol of the class \(S^{m,\psi}_0(\mathbb{R}^n)\).

Note that \(S^{m,\psi}_0(\mathbb{R}^n) \subset S^{m,\psi}(\mathbb{R}^n)\). For \(q \in S^{m,\psi}_0(\mathbb{R}^n)\), hence also for \(q \in S^{m,\psi}(\mathbb{R}^n)\), we can define on \(S(\mathbb{R}^n)\) the pseudo-differential operator \(q(x,D)\) by

\[ q(x,D)u(x) := (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix \cdot \xi} q(x,\xi) \hat{u}(\xi) d\xi \]

and we denote the classes of these operators by \(\Psi^{m,\psi}_0(\mathbb{R}^n)\) and \(\Psi^{m,\psi}(\mathbb{R}^n)\), respectively.
Theorem 1.3. Let \( q \in S^m_{\rho, \psi}(\mathbb{R}^n) \) then \( q(x, D) \) maps \( S(\mathbb{R}^n) \) continuously into itself.

Let \( \psi : \mathbb{R}^n \to \mathbb{R} \) be a fixed continuous negative definite function. For \( s \in \mathbb{R} \) and \( u \in S(\mathbb{R}^n) \) (or \( u \in S'(\mathbb{R}^n) \)) we define the norm
\[
\|u\|_{\psi,s}^2 = \| (1 + \psi(D))^\frac{1}{2} u \|_0^2 = \int_{\mathbb{R}^n} (1 + \psi(s)) |\hat{u}(\xi)|^2 \, d\xi.
\]

The space \( H^{\psi,s}(\mathbb{R}^n) \) is defined as
\[
H^{\psi,s}(\mathbb{R}^n) := \{ u \in S'(\mathbb{R}^n) : \|u\|_{\psi,s} < \infty \}.
\]

The scale \( H^{\psi,s}(\mathbb{R}^n) \), \( s \in \mathbb{R}^n \), and more general spaces have been systematically investigated in [3,4], see also [12]. In particular we know that if for some \( \rho_1 > 0 \) and \( c_1 > 0 \) the estimate \( \psi(\xi) \geq c_1 |\xi|^\rho_1 \) holds for all \( \xi \in \mathbb{R}^n \), \( |\xi| \geq R, R \geq 0 \), then the space \( H^{\psi,s}(\mathbb{R}^n) \) is continuously embedded into \( C_\infty(\mathbb{R}^n) \) provided \( s > \frac{n}{2\rho_1} \).

Theorem 1.4. Let \( q \in S^m_{\rho, \psi}(\mathbb{R}^n) \) and let \( q(x, D) \) be the corresponding pseudo-differential operator. For all \( s \in \mathbb{R} \) the operator \( q(x, D) \) maps the space \( H^{\psi,m+s}(\mathbb{R}^n) \) continuously into the space \( H^{\psi,s}(\mathbb{R}^n) \), and for all \( u \in H^{\psi,m+s}(\mathbb{R}^n) \) we have the estimate
\[
\|q(x, D)u\|_{\psi,s} \leq c \|u\|_{\psi,m+s}.
\]

On \( S(\mathbb{R}^n) \) we may define the bilinear form
\[
B(u, v) := (q(x, D)u, v)_{0}, \quad q \in S^{m, \psi}_{\rho}(\mathbb{R}^n).
\]

Theorem 1.5. Let \( q \in S^{m, \psi}_{\rho}(\mathbb{R}^n) \) be real valued and \( m > 0 \). It follows that
\[
|B(u, v)| \leq c \|u\|_{\psi, \frac{m}{2}} \|v\|_{\psi, \frac{m}{2}}
\]
holds for all \( u, v \in S(\mathbb{R}^n) \). Hence the bilinear form \( B \) has a continuous extension onto \( H^{\psi, \frac{m}{2}}(\mathbb{R}^n) \). If in addition for all \( x \in \mathbb{R}^n \)
\[
q(x, \xi) \geq \delta_0 (1 + \psi(\xi))^m \quad \text{for} \quad |\xi| \geq R
\]
with some \( \delta_0 > 0 \) and \( R \geq 0 \), and
\[
\lim_{|\xi| \to \infty} \psi(\xi) = \infty
\]
holds, then we have for all \( u \in H^{\psi, \frac{m}{2}}(\mathbb{R}^n) \) the Gårding inequality
\[
ReB(u, u) \geq \frac{\delta_0}{2} \|u\|^2_{\frac{m}{2}} - \lambda_0 \|u\|^2_{0}.
\]
Furthermore we have

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Theorem 1.6. If we assume (1) and (2) then for \( s > -m \) we have
\[
\frac{\sigma_0}{2} \|u\|_{\psi, m+s} \leq \|q(x, D)u\|_{\psi, s}^2 + \|u\|_{\psi, m+s}^2
\]
for \( q \in S^m_\rho(\mathbb{R}^n) \) real-valued and all \( u \in H^{\psi, s+m}(\mathbb{R}^n) \).

From Theorem 1.5 and 1.6 one may deduce the following regularity result:

Theorem 1.7. Let \( q \in S^m_\rho(\mathbb{R}^n) \) be as in Theorem 1.6, \( m \geq 1 \). Further suppose that for \( f \in H^{\psi, s}(\mathbb{R}^n), s \geq 0 \), there exists \( u \in H^{\psi, s+m}(\mathbb{R}^n) \) such that
\[
B(u, \phi) = (f, \phi)_{L^2}
\]
holds for all \( \phi \in H^{\psi, s+m}(\mathbb{R}^n) \) (or \( \phi \in S(\mathbb{R}^n) \)). Then \( u \) belongs already to the space \( H^{\psi, s+m}(\mathbb{R}^n) \).

So far we have used properties of symbols to establish mapping properties and estimates for operators. The real power of a symbolic calculus is that it reduces calculations for operators to calculations for symbols. The following result is most important for us

Theorem 1.8. Let \( \psi \in \Lambda \). For \( q_1 \in S^{m_1}_{\rho}(\mathbb{R}^n) \) and \( q_2 \in S^{m_2}_{\rho}(\mathbb{R}^n) \) the symbol \( q \) of the operator \( q(x, D) := q_1(x, D) \circ q_2(x, D) \) is given by
\[
q(x, \xi) = q_1(x, \xi) \cdot q_2(x, \xi) + \sum_{j=1}^{n} \partial_{\xi_j} q_1(x, \xi) D_{x_j} q_2(x, \xi) + q_{r_1}(x, \xi)
\]
with \( q_{r_1} \in S^{m_1+m_2-2\psi}_{\rho}(\mathbb{R}^n) \).

Remark 1.9. An easy calculation yields \( q_1 \cdot q_2 \in S^{m_1+m_2}_{\rho}(\mathbb{R}^n), \partial_{\xi_j} q_1 \in S^{m_1-1}_{\rho}(\mathbb{R}^n) \), and \( D_{x_j} q_2 \in S^{m_2}_{\rho}(\mathbb{R}^n) \). Hence the second term on the right hand side in (3) belongs to \( S^{m_1+m_2-1}_{\rho}(\mathbb{R}^n) \).

2. The formal background of our proof that \( -p(x, D) \) generates a Feller semigroup

The proof that \( -p(x, D) \) as described in the introduction, see also below, extends to a generator of a Feller semigroup depends on various estimates which might be different for different operators. However, once these estimates are established we only need to apply a piece of “soft” analysis. In this section we discuss this part of the proof, i.e., we will assume all crucial estimates hold. Let \( f : \mathbb{R}^n \times [0, \infty) \rightarrow \mathbb{R} \) be an arbitrarily often differentiable function such that for \( y \in \mathbb{R}^n \) fixed the function \( s \rightarrow f(y, s) \) is a Bernstein function. Moreover we assume
\[
\inf_{y \in \mathbb{R}^n} f(y, s) \geq f_0(s) \quad \text{for all} \quad s \in [0, \infty)
\]
as well as
\[
\sup_{y \in \mathbb{R}^n} f(y, s) \leq f_1(s) \quad \text{for all } s \in [0, \infty)
\]
(5)
where \( f_0 \) and \( f_1 \) are Bernstein functions. For a given real-valued negative definite symbol \( q(x, \xi) \) it follows that
\[
p(y; x, \xi) := f(y, q(x, \xi))
\]
give rise to a further negative definite symbol by defining
\[
p(x, \xi) := p(x; x, \xi)
\]
(6)
In case where \( q(x, \xi) \) is comparable with a fixed continuous negative definite function \( \psi \), i.e.,
\[
0 < c_0 \leq \frac{q(x, \xi)}{\psi(\xi)} \leq c_1, \quad c_1 \geq 1,
\]
(7)
for all \( x \in \mathbb{R}^n \) and \( \xi \in \mathbb{R}^n \), we find using [11, Lemma 3.9.34.B]
\[
p(x, \xi) \leq f(y_1, q(x, \xi)) \leq c_1 f_1(\psi(\xi))
\]
and we define
\[
\psi_1(\xi) := c_1 f_1(\psi(\xi)).
\]
(8)
Moreover it holds
\[
p(x, \xi) \geq f(y_0, q(x, \xi)) \geq c_0 f_0(\psi(\xi))
\]
and we set
\[
\psi_0(\xi) := c_0 f_0(\psi(\xi)).
\]
(9)
Clearly, \( \psi_0 \) and \( \psi_1 \) are continuous negative definite functions. Later on we assume that for \(|\xi|\) large
\[
\psi(\xi) \geq \tilde{c}_1 |\xi|^{\rho_1}, \quad \tilde{c}_1 > 0 \quad \text{and} \quad \rho_1 > 0
\]
holds as well as
\[
f(y_0, s) \geq \tilde{c}_0 s^{\rho_0}, \quad \tilde{c}_0 > 0 \quad \text{and} \quad \rho_0 > 0.
\]
(10)
This implies for \(|\xi|\) large that
\[
\psi_0(\xi) \geq \tilde{c}_2 |\xi|^{\rho_0 \rho_1}, \quad \tilde{c}_2 > 0,
\]
(12)
holds. Since \( \psi_0(\xi) \leq \psi_1(\xi) \) we have
\[
\mathcal{H}^{\psi_1,1}(\mathbb{R}^n) \hookrightarrow \mathcal{H}^{\psi_0,1}(\mathbb{R}^n).
\]
We add the assumption that there exists \( 0 < \sigma < \frac{1}{2} \) such that
\[
(1 + \psi_1)^{\frac{1}{2}} \in S^{1+\sigma,\psi_0}_\rho(\mathbb{R}^n).
\]
(13)
This will imply that 
\[ H^{\psi_0, m(1+\sigma)}(\mathbb{R}^n) \hookrightarrow H^{\psi_1, m}(\mathbb{R}^n) \] 
holds for \( m \geq 0 \). Further, (13) implies that if \( p_1(x, \xi) \) is any symbol belonging to \( S^m_p(\mathbb{R}^n) \) then it also belongs to \( S^{m(1+\sigma)}_p(\mathbb{R}^n) \) which follows from
\[
|\partial_\xi^\alpha \partial_x^\beta p_1(x, \xi)| \leq c_{\alpha, \beta}(1 + \psi_1(\xi)) \frac{m - \rho(m)}{2} \\
\leq \tilde{c}_{\alpha, \beta}(1 + \psi_0(\xi)) \frac{m - \rho(m)(1+\sigma)}{2} \\
\leq \tilde{c}_{\alpha, \beta}(1 + \psi_0(\xi)) \frac{(1+\sigma)m - \rho(m)}{2}.
\]

The pseudo-differential operator \( q(x, D) \) has the symbol \( q \in S^2_n(\mathbb{R}^n) \). We assume that the pseudo-differential operator \( p(x, D) \), defined on \( S(\mathbb{R}^n) \) by
\[
p(x, D)u(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix \cdot \xi} p(x, \xi) \hat{u}(\xi) \, d\xi
\]
has a symbol \( p \in S^{2+\tau_1, \psi_1}(\mathbb{R}^n) \) for some appropriate \( \tau_1 \geq 0 \). This implies together with (13) that the operator \( p(x, D) \) is continuous from \( H^{\psi_0, 2+\tau_1, 2\sigma + \tau_1, \sigma + s}(\mathbb{R}^n) \) to \( H^{\psi_0, s}(\mathbb{R}^n) \), in particular it is continuous from \( H^{\psi_0, 1}(\mathbb{R}^n) \) to \( H^{\psi_0, -1-\tau_1-2\sigma-\tau_1, \sigma}(\mathbb{R}^n) \).

With \( p(x, D) \) we can associate the bilinear form
\[
B(u, v) := (p(x, D)u, v)_0, \quad u, v \in S(\mathbb{R}^n).
\]
Assuming the estimate
\[
|B(u, v)| \leq \kappa \|u\|_{\psi_1, 1} \|v\|_{\psi_1, 1}, \quad \kappa \geq 0,
\]
to hold for all \( u, v \in S(\mathbb{R}^n) \), we may extend \( B \) to a continuous bilinear form on \( H^{\psi_1, 1}(\mathbb{R}^n) \). This extension is again denoted by \( B \). For \( u \in H^{\psi_1, 1}(\mathbb{R}^n) \) we assume in addition
\[
B(u, u) \geq \gamma \|u\|_{\psi_1, 1}^2 - \lambda_0 \|u\|_0^2, \quad f \lambda_0 \geq 0, \quad \gamma > 0. \tag{15}
\]
Following ideas from I. S. Louhivaara and Ch. Simader, [18,19], we consider an intermediate space associated with
\[
B_{\lambda_0}(u, v) := B(u, v) + \lambda_0 (u, v)_0,
\]

namely the space \( H^{F_{\lambda_0}}(\mathbb{R}^n) \) defined as a completion of \( S(\mathbb{R}^n) \) (or \( H^{\psi_1, 1}(\mathbb{R}^n) \)) with respect to the scalar product \( B_{\lambda_0} \). Obviously we have
\[
H^{\psi_1, 1}(\mathbb{R}^n) \hookrightarrow H^{F_{\lambda_0}}(\mathbb{R}^n) \hookrightarrow H^{\psi_0, 1}(\mathbb{R}^n) \tag{16}
\]
in the sense of continuous embeddings. Moreover, by the Lax-Milgram theorem, for every \( g \in (H^{p,0}(\mathbb{R}^n))^* \) exists a unique element \( u \in H^{p,0}(\mathbb{R}^n) \) satisfying
\[
B_{\lambda_0}(u, v) = \langle g, v \rangle
\]
for all \( v \in H^{p,0}(\mathbb{R}^n) \). This element we call the variational solution to the equation \( p(x, D)u + \lambda_0 u = g \).

From (16) we derive
\[
H^{p,0, -1}(\mathbb{R}^n) = (H^{p,0,1}(\mathbb{R}^n))^* \hookrightarrow (H^{p,0}(\mathbb{R}^n))^*,
\]
hence for \( g \in H^{p,0, -1}(\mathbb{R}^n) \) there exists a unique \( u \in H^{p,0}(\mathbb{R}^n) \) satisfying (17). We claim now that for every \( g \in H^{p,0, -1}(\mathbb{R}^n) \) there exists a unique \( u \in H^{p,0,1}(\mathbb{R}^n) \) such that
\[
p_{\lambda_0}(x, D)u = p(x, D)u + \lambda_0 u = g
\]
holds. Denote by \( u \in H^{p,0}(\mathbb{R}^n) \) the unique solution to (17) for \( g \in H^{p,0, -1}(\mathbb{R}^n) \) given and take a sequence \( (u_k)_{k \in \mathbb{N}}, u_k \in S(\mathbb{R}^n), \) converging in \( H^{p,0}(\mathbb{R}^n) \) to \( u \). It follows from
\[
(p_{\lambda_0}(x, D)u_k, v)_0 = B_{\lambda_0}(u_k, v), \quad v \in S(\mathbb{R}^n),
\]
and the continuity of \( p_{\lambda_0}(x, D) \) from \( H^{p,0,1}(\mathbb{R}^n) \) into \( H^{p,0, -1}(\mathbb{R}^n) \) that for \( k \to \infty \)
\[
\langle p_{\lambda_0}(x, D)u, v \rangle = B_{\lambda_0}(u, v) = \langle g, v \rangle
\]
for all \( v \in S(\mathbb{R}^n) \). Thus \( p_{\lambda_0}(x, D)u = g \). The uniqueness follows of course once again from (15).

In order to get more regularity for variational solutions or equivalently for solutions to (18) we assume that for \( \lambda \geq \lambda_0 \) the function \( p_{\lambda}^{-1}(x, \xi) := \frac{1}{p(x, \xi + \lambda)} \) belongs to \( S_{p}^{1+\tau_1, \psi_0}(\mathbb{R}^n) \) for some \( \tau_0 > 0 \). In this case we can prove

**Theorem 2.1.** Let \( p(x, \xi) \) be given by (6) where we assume for \( q \) condition (7) and for \( f \) we require (4), (5) to hold. In addition we suppose that \( p \in S_{p}^{2+\tau_1, \psi_0}(\mathbb{R}^n) \subset S_{p}^{2+\tau_1+2\tau_0+2\tau_1, \psi_0}(\mathbb{R}^n) \) and \( p_{\lambda}^{-1} \in S_{p}^{2+\tau_0, \psi_0}(\mathbb{R}^n), \tau_1 + \tau_0 + 2\sigma + \tau_1 \sigma < 1 \). Let \( u \in H^{p,0}(\mathbb{R}^n) \subset H^{p,0,1}(\mathbb{R}^n) \) be the solution to (18) for \( g \in H^{p,0,k}(\mathbb{R}^n) \), \( k \geq 0 \). Then it follows that \( u \in H^{p,0,2+k-\tau_0}(\mathbb{R}^n) \).

**Proof.** From Theorem 1.8 it follows that
\[
p_{\lambda_0}^{-1}(x, D) \circ p_{\lambda_0}(x, D) = id + r(x, D)
\]
with \( r \in S_{0}^{1+\tau_1+\tau_0+2\sigma+\tau_1, \psi_0}(\mathbb{R}^n) \). Since \( p_{\lambda_0}(x, D)u = g \) we deduce from (19) that
\[
u = p_{\lambda_0}^{-1}(x, D) \circ p_{\lambda_0}(x, D)u = r(x, D)u
\]
\[
= p_{\lambda_0}^{-1}(x, D)g - r(x, D)u.
\]
Now, \( p_{\lambda_0}^{-1}(x, D)g \in H^{p,0,k+2-\tau_0}(\mathbb{R}^n) \) and \( r(x, D)u \in H^{p,0,2-\tau_1-\tau_0-2\sigma-\tau_1 \sigma}(\mathbb{R}^n) \) implying that \( u \in H^{p,0,t}(\mathbb{R}^n) \) for \( t = (k + 2 - \tau_0) \wedge (2 - \tau_1 - \tau_0 - 2\sigma - \tau_1 \sigma) > 1 \). With a finite number of iterations we arrive at \( u \in H^{p,0,2+k-\tau_0}(\mathbb{R}^n) \).
Remark 2.2.  From $\tau_1 + \tau_0 + 2\sigma + \tau_1\sigma < 1$ the necessary condition $\sigma < \frac{1}{2}$ follows.

Corollary 2.3.  In the situation of Theorem 2.1, if $2 + k - \tau_0 > \frac{n}{2\rho_{\max}}$, compare (12), then $u \in C_{\infty}(\mathbb{R}^n)$.

Finally we can collect all preparatory material to prove

Theorem 2.4.  Let $f: \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ be an arbitrarily often differentiable function such that for $y \in \mathbb{R}^n$ fixed, the function $s \to f(y, s)$ is a Bernstein function.  Moreover assume (4), (5), and (11).  In addition let $\psi : \mathbb{R}^n \to \mathbb{R}$ be a continuous negative definite function in the class $\Lambda$ which satisfies in addition (10).  For an elliptic symbol $q \in S^2_{\rho, \psi}(\mathbb{R}^n)$ satisfying (7) we define $p(x, \xi)$ by (6).  For $\psi_1$ and $\psi_2$ defined by (8) and (9), respectively we assume (14).  Suppose that $p \in S^2_{\rho, \psi_1}(\mathbb{R}^n)$ and $\frac{1}{p(x, \xi)} \in S^2_{\rho, \psi_2}(\mathbb{R}^n)$.  If $\tau_1 + \tau_0 + \sigma(2 + \tau_1) < 1$, $\sigma$ as in (14), then $-p(x, D)$ extends to a generator of a Feller semigroup on $C_{\infty}(\mathbb{R}^n)$.

Proof.  We want to apply the Hille-Yosida-Ray theorem, compare [11, Theorem 4.5.3].  We know that $p(x, D)$ maps $H^{\psi_2}(\mathbb{R}^n)$ into $H^{\psi_1}(\mathbb{R}^n)$.  Hence if $k > \frac{n}{2\rho_{\max}}$ the operator $(-p(x, D), H^{\psi_2}(\mathbb{R}^n))$ is densely defined on $C_{\infty}(\mathbb{R}^n)$ with range in $C_{\infty}(\mathbb{R}^n)$.  That $-p(x, D)$ satisfies the positive maximum principle on $H^{\psi_2}(\mathbb{R}^n)$ follows from [12, Theorem 2.6.1].  Now, for $\lambda \geq \lambda_0$ we know that for $g \in H^{\psi_2}(\mathbb{R}^n)$ there exists a unique solution to $p(x, D)u = g$ belonging to $H^{\psi_2}(\mathbb{R}^n)$.

3. Some concrete examples

The first part of this section will consider the work W. Hoh has done on pseudo-differential operators with variable order of differentiation.  We will consider the case where the Bernstein function $s \to f(s)$ is substituted by $(x, s) \to s^{r(x)}$ with $r : \mathbb{R}^n \to \mathbb{R}$ being a continuous function such that $0 \leq r(x) \leq 1$ holds.  Let $q : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ be a continuous function such that $\xi \to q(x, \xi)$ is a continuous negative definite function.  It then follows that $\xi \to q(x, \xi)^{r(x)}$ is once again a continuous negative definite function implying that the pseudo-differential operator

$$A u(x) := -(2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix \xi} q(x, \xi)^{r(x)} \hat{u}(\xi) d\xi$$

is a candidate for a generator of a Feller semigroup.  We now meet Hoh’s result:

Theorem 3.1.  Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be a fixed continuous negative definite function such that its Lévy measure has a compact support and that

$$\psi(\xi) \geq c_0 |\xi|^r, \quad |\xi| \text{ large and } r > 0,$$
holds. Let \( q \in S^{2,\psi}_\mu(\mathbb{R}^n) \) be a real-valued negative definite symbol which is elliptic, i.e., we have

\[
q(x, \xi) \geq \delta_0 (1 + \psi(\xi)).
\]

Further let \( m : \mathbb{R}^n \to (0, 1] \) be an element in \( C_0^\infty(\mathbb{R}^n) \) satisfying

\[
M - \mu < \frac{1}{2}
\]

where \( M := \sup m(x) \) and \( 0 < \mu := \inf m(x) \). Consider the symbol

\[
(x, \xi) \mapsto p(x, \xi) := q(x, \xi)^m(x)
\]

which has the property that \( \xi \mapsto p(x, \xi) \) is a continuous negative definite function. The operator

\[
-p(x, D)u(x) := -(2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix \cdot \xi} p(x, \xi) \hat{u}(\xi) d\xi
\]

maps \( C_0^\infty(\mathbb{R}^n) \) into \( C_\infty(\mathbb{R}^n) \), is closeable in \( C_\infty(\mathbb{R}^n) \) and its closure is a generator of a Feller semigroup.

For a proof see W. Hoh [7], compare also [6].

We are now going to consider a further example. First note that the function \( s \mapsto \sqrt{s}(1 - e^{-4\sqrt{s}}) \) is a Bernstein function. Hence, using [11, Corollary 3.9.36], it follows that for \( 0 \leq \alpha \leq 1 \) the function \( s \mapsto \sqrt{s}(1 - e^{-4s^{\alpha/2}}) \) is also a Bernstein function. Thus, given a negative definite symbol \( q \in S^{2,\psi}_\mu(\mathbb{R}^n) \) we may consider the new symbol

\[
p(x, \xi) = (1 + q(x, \xi))^{\frac{\alpha(x)}{2}} \left( 1 - e^{-4(1+q(x,\xi))^{\frac{\alpha(x)}{2}}} \right)
\]

for \( \alpha(\cdot) \) being an appropriate function.

**Lemma 3.2.** Let \( q \in S^{2,\psi}_\mu(\mathbb{R}^n) \) be a real-valued negative definite symbol which is elliptic, i.e.,

\[
q(x, \xi) \geq \delta_0 (1 + \psi(\xi)).
\]

Also let \( \alpha(\cdot) : \mathbb{R}^n \to (0, 1] \) be an element in \( C_0^\infty(\mathbb{R}^n) \) satisfying

\[
m - \mu < \frac{1}{2}
\]

where \( m = \sup \frac{\alpha(x)}{2} \) and \( \mu = \inf \frac{\alpha(x)}{2} > 0 \).

Now if we let \( p(x, \xi) = (1 + q(x, \xi))^{\frac{\alpha(x)}{2}} \left( 1 - e^{-4(1+q(x,\xi))^{\frac{\alpha(x)}{2}}} \right) \), then we have for all \( \epsilon > 0 \) the estimates

\[
|\partial_\xi^\alpha \partial_x^\beta p(x, \xi)| \leq c_{\alpha,\beta,\epsilon} p(x, \xi) (1 + \psi(\xi))^{-\frac{\epsilon^2(1+q(x,\xi))^{\frac{\alpha(x)}{2}}}{2}} \tag{20}
\]

i.e., \( p \in S^{2m+\epsilon,\psi}_\mu(\mathbb{R}^n) \).
Proof. We have to estimate
\[ \partial_{\xi}^{\alpha} \partial_{x}^{\beta} p(x, \xi) = \partial_{\xi}^{\alpha} \partial_{x}^{\beta} \left( (1 + q(x, \xi))^{\frac{\alpha(x)}{2}} \left( 1 - e^{-4(1+q(x,\xi))^{\frac{\alpha(x)}{2}}} \right) \right) \]
\[ = \partial_{\xi}^{\alpha} \partial_{x}^{\beta} \left( e^{\frac{\alpha(x)}{2} \log(1+q(x,\xi))} \left( 1 - e^{-4(1+q(x,\xi))^{\frac{\alpha(x)}{2}}} \right) \right). \]

Using \[11, (2.19)\] we get
\[ \partial_{\xi}^{\alpha} \partial_{x}^{\beta} \left( e^{\frac{\alpha(x)}{2} \log(1+q(x,\xi))} \left( 1 - e^{-4(1+q(x,\xi))^{\frac{\alpha(x)}{2}}} \right) \right) \]
\[ = \sum_{\alpha' \leq \alpha, \beta' \leq \beta} \left( \alpha' \beta' \right) (\partial_{\xi}^{\alpha'} \partial_{x}^{\beta'} e^{\frac{\alpha(x)}{2} \log(1+q(x,\xi))}) \]
\[ \times \left( 1 - e^{-4(1+q(x,\xi))^{\frac{\alpha(x)}{2}}} \right). \] (21)

First consider
\[ ||(\partial_{\xi}^{\alpha'} \partial_{x}^{\beta'} e^{\frac{\alpha(x)}{2} \log(1+q(x,\xi))}||. \]

By \[11, (2.28)\] with \( l = \alpha' + \beta' \) we get
\[ ||(\partial_{\xi}^{\alpha'} \partial_{x}^{\beta'} e^{\frac{\alpha(x)}{2} \log(1+q(x,\xi))}) || \leq e^{\frac{\alpha(x)}{2} \log(1+q(x,\xi))} \sum_{\alpha'^{1} + \ldots + \alpha'^{l'} = \alpha'} \prod_{j=1}^{l'} q_{\alpha'^{j},\beta'^{j}}(x, \xi). \]
(22)

where
\[ q_{\alpha',\beta'} (x, \xi) = \partial_{\xi}^{\alpha'} \partial_{x}^{\beta'} \left( \frac{\alpha(x)}{2} \log(1 + q(x,\xi)) \right) \]
\[ = \sum_{\beta'^{1} + \ldots + \beta'^{l'} = \beta'} \left( \frac{\beta'^{1}}{\beta'} \right) (\partial_{x}^{\beta'^{1} - \beta'} \alpha(x) \partial_{\xi}^{\alpha') \partial_{x}^{\beta'} \log(1 + q(x,\xi)). \]

Now, using \[11, (2.26)\] with \( k = |\alpha'| + |\beta'| > 0 \) we get
\[ \partial_{\xi}^{\alpha'} \partial_{x}^{\beta'} \log(1 + q(x,\xi)) = \sum_{\tilde{\alpha}^{1} + \ldots + \tilde{\alpha}^{k} = \alpha'^{1} + \ldots + \alpha'^{l'}} \prod_{i=1}^{k} \frac{\partial_{\xi}^{\alpha'^{i}} \partial_{x}^{\beta'} \log(1 + q(x,\xi))}{(1 + q(x,\xi)).} \]
Since we assume that \(q(x, \xi)\) is an elliptic symbol in \(S^0_{\rho}(\mathbb{R}^n)\), we get
\[
\left| \partial_x^{\alpha'} \partial_{\xi}^{\beta'} \log(1 + q(x, \xi)) \right| \leq c_{\alpha', \beta'} \left( 1 + \psi(\xi) \right)^{-\frac{\omega(\alpha')}{2}}, \quad \alpha' \neq 0
\]
\[
\left( 1 + \psi(\xi) \right)^{\frac{\omega(\alpha')}{2}}, \quad \alpha' = 0.
\]
Putting (22) and (23) together we get
\[
\left| \partial_x^{\alpha'} \partial_{\xi}^{\beta'} \right| \leq c_{\alpha', \beta'} \left( 1 + \psi(\xi) \right)^{-\frac{\omega(\alpha')}{2}}.
\]
where we used the subadditivity of \(\rho\). We always have
\[
\left| \log(1 + q(x, \xi)) \right| \leq c_{\epsilon} \left( 1 + \psi(\xi) \right)^{\frac{\epsilon}{2}}.
\]
It follows for \(\alpha \in C_0^\infty(\mathbb{R}^n)\) that
\[
\left| q_{\alpha', \beta'}(x, \xi) \right| \leq c_{\alpha', \beta', \epsilon} \left\{ \begin{array}{ll}
1 + \psi(\xi) \leq 0, & \\
\left( 1 + \psi(\xi) \right)^{\frac{\omega(\alpha')}{2}}, & \alpha' \neq 0. \end{array} \right.
\]
Putting (22) and (23) together we get
\[
\left| \partial_x^{\alpha'} \partial_{\xi}^{\beta'} \left( 1 - e^{-4(1+q(x, \xi)) \frac{\alpha(\epsilon)}{2}} \right) \right| \leq c_{\alpha', \beta', \alpha, \beta, \epsilon} \left( 1 - e^{-4(1+q(x, \xi)) \frac{\alpha(\epsilon)}{2}} \right) \left( 1 + \psi(\xi) \right)^{-\frac{\omega(\alpha')}{2}}.
\]
For the desired result we need
\[
\left| \partial_x^{\alpha' - \alpha} \partial_{\xi}^{\beta' - \beta} \left( 1 - e^{-4(1+q(x, \xi)) \frac{\alpha(\epsilon)}{2}} \right) \right| \leq c_{\alpha', \beta', \alpha, \beta, \epsilon} \left( 1 - e^{-4(1+q(x, \xi)) \frac{\alpha(\epsilon)}{2}} \right) \left( 1 + \psi(\xi) \right)^{-\frac{\omega(\alpha')}{2}}.
\]
When \(\alpha - \alpha' = 0\) and \(\beta - \beta' = 0\) there is nothing to prove.
Otherwise, by [11, (2.28)] with \(l_2 = |\alpha - \alpha'| + |\beta - \beta'|\), we get
\[
\left| \partial_x^{\alpha' - \alpha} \partial_{\xi}^{\beta' - \beta} \left( 1 - e^{-4(1+q(x, \xi)) \frac{\alpha(\epsilon)}{2}} \right) \right| \leq e^{-4(1+q(x, \xi)) \frac{\alpha(\epsilon)}{2}} \sum_{j=1}^{l_2} c_{\{\alpha - \alpha'\}, \{\beta - \beta'\}} \prod_{j=1}^{l_2} q_{\alpha - \alpha' \beta' \beta'}(x, \xi),
\]
where the sum is such that
\[
(\alpha - \alpha')^1 + \cdots + (\alpha - \alpha')^{l_2} = (\alpha - \alpha'),
\]
\[
(\beta - \beta')^1 + \cdots + (\beta - \beta')^{l_2} = (\beta - \beta'),
\]
\(l_2 = 1, \ldots, l_2\),
and where
\[
q_{\alpha - \alpha' \beta' \beta'}(x, \xi) = \partial_x^{\alpha(\epsilon)/2} \partial_{\xi}^{\beta(\epsilon)/2} \left( 4(1 + q(x, \xi)) \frac{\alpha(\epsilon)}{2} \right).
\]
Since $q(x, \xi)$ is an elliptic symbol in the class $S^2_{\rho} \psi(\mathbb{R}^n)$ we have the estimate

$$|q_{(\alpha-\alpha')j(\beta-\beta')j}(x, \xi)| \leq \tilde{L}(1 + q(x, \xi)) \quad \text{for all} \quad (\alpha - \alpha')^j, (\beta - \beta')^j \in \mathbb{N}_0^n,$$

where $\tilde{L}(\lambda)$ is a suitable polynomial $\geq 0$ which might depend on $(\alpha - \alpha')^j$ and $(\beta - \beta')^j$.

Now returning to (25) we get

$$|\partial_\xi^{(\alpha-\alpha')j}\partial_\xi^{(\beta-\beta')j} \left(1 - e^{-4(1 + q(x, \xi)) \frac{\alpha(x)}{2}} \right)|$$

$$\leq \tilde{L}(1 + q(x, \xi)) e^{-4(1 + q(x, \xi)) \frac{\alpha(x)}{2}}$$

$$= \frac{4(1 + q(x, \xi)) \alpha(x)}{1 + 4(1 + q(x, \xi)) \frac{\alpha(x)}{2}} \frac{1 + 4(1 + q(x, \xi)) \alpha(x)}{4(1 + q(x, \xi)) \frac{\alpha(x)}{2}} \tilde{L}(1 + q(x, \xi)) e^{-4(1 + q(x, \xi)) \frac{\alpha(x)}{2}}$$

$$\times (1 + \psi(\xi)) \frac{\alpha(x)}{4(1 + q(x, \xi)) \frac{\alpha(x)}{2}}(1 + \psi(\xi)) \frac{\alpha(x)}{4(1 + q(x, \xi)) \frac{\alpha(x)}{2}}$$

$$\leq \frac{4(1 + q(x, \xi)) \alpha(x)}{1 + 4(1 + q(x, \xi)) \frac{\alpha(x)}{2}} (1 + \psi(\xi)) \frac{\alpha(x)}{4(1 + q(x, \xi)) \frac{\alpha(x)}{2}} c_0.$$

Now using [12, (2.7)], i.e., for all $a \geq 0$ and $t \geq 0$ the estimate

$$\frac{at}{1 + at} \leq 1 - e^{-at},$$

we get

$$|\partial_\xi^{(\alpha - \alpha')j}\partial_\xi^{(\beta - \beta')j} \left(1 - e^{-4(1 + q(x, \xi)) \frac{\alpha(x)}{2}} \right)|$$

$$\leq c_0 \left(1 - e^{-4(1 + q(x, \xi)) \frac{\alpha(x)}{2}} \right)(1 + \psi(\xi)) \frac{\alpha(x)}{4(1 + q(x, \xi)) \frac{\alpha(x)}{2}}.$$ (26)

Substituting (24) and (26) into (21)

$$|\partial^\alpha \left(e^{\frac{\alpha(x)}{2} \log(1 + q(x, \xi)) \left(1 - e^{-4(1 + q(x, \xi)) \frac{\alpha(x)}{2}} \right)} \right)|$$

$$\leq \sum_{\alpha' \leq \alpha} \sum_{\beta' \leq \beta} \binom{\alpha}{\alpha'} \binom{\beta}{\beta'} c_{\alpha', \beta', e} \frac{\alpha(x)}{4(1 + q(x, \xi)) \frac{\alpha(x)}{2}}$$

$$\times (1 + \psi(\xi)) \frac{\alpha(x)}{4(1 + q(x, \xi)) \frac{\alpha(x)}{2}} \left(1 - e^{-4(1 + q(x, \xi)) \frac{\alpha(x)}{2}} \right)(1 + \psi(\xi)) \frac{\alpha(x)}{4(1 + q(x, \xi)) \frac{\alpha(x)}{2}}$$

$$\leq c_{\alpha, \beta, e} \frac{\alpha(x)}{4(1 + q(x, \xi)) \frac{\alpha(x)}{2}} \left(1 - e^{-4(1 + q(x, \xi)) \frac{\alpha(x)}{2}} \right)$$

$$\times (1 + \psi(\xi)) \frac{\alpha(x)}{4(1 + q(x, \xi)) \frac{\alpha(x)}{2}}$$

$$\leq c_{\alpha, \beta, e} p(x, \xi)(1 + \psi(\xi)) \frac{\alpha(x)}{4(1 + q(x, \xi)) \frac{\alpha(x)}{2}}.$$
The proof now follows from the estimate $p(x, \xi) \leq (1 + \psi(\xi))^m$.  

**Lemma 3.3.** The function $p^{-1}_\lambda(x, \xi) = \frac{1}{p(x, \xi)^\lambda}$ belongs to the class $S_{-2\mu+\epsilon, \psi}^\rho(R^n)$.

**Proof.** Using \cite[(2.27)]{11} we find with $l = |\alpha| + |\beta|$ that

$$|\partial^\alpha_x \partial^\beta_\xi p^{-1}_\lambda(x, \xi)| \leq \frac{1}{p(x, \xi)^\lambda} \sum_{\alpha_1 + \cdots + \alpha' = \alpha} e_{\alpha, \beta} \prod_{j=1}^l \left| \frac{\partial^\alpha_x \partial^\beta_\xi p_\lambda(x, \xi)}{p_\lambda(x, \xi)} \right|.$$  

For any $\epsilon > 0$ we find using (20)

$$|\partial^\alpha_x \partial^\beta_\xi p^{-1}_\lambda(x, \xi)| \leq \tilde{c}_{\alpha, \beta, \epsilon} (1 + \psi(\xi))^{\frac{|\alpha|+|\beta|}{\epsilon}}$$

and the ellipticity assumption of $p(x, \xi)$ together with the subadditivity of $\rho$ yields

$$|\partial^\alpha_x \partial^\beta_\xi p^{-1}_\lambda(x, \xi)| \leq \tilde{c}_{\alpha, \beta, \epsilon} (1 + \psi(\xi))^{-\mu} (1 + \psi(\xi))^{-\frac{1}{2}}$$

which proves the lemma. 

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**References**


K. P. Evans/N. Jacob

Feller semigroups obtained by variable order subordination


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