Fundamental Groups of Some Special Quadric Arrangements

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ABSTRACT

Continuing our work on the fundamental groups of conic-line arrangements [3], we obtain presentations of fundamental groups of the complements of three families of quadric arrangements in \( \mathbb{P}^2 \). The first arrangement is a union of \( n \) conics, which are tangent to each other at two common points. The second arrangement is composed of \( n \) quadrics which are tangent to each other at one common point. The third arrangement is composed of \( n \) quadrics, \( n - 1 \) of them are tangent to the \( n \)th one and each one of the \( n - 1 \) quadrics is transversal to the other \( n - 2 \) ones.

Key words: fundamental groups, complement of curve, conic arrangement.

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Introduction

The aim of the present article is the computation of the fundamental groups to complements of some quadric arrangements in \( \mathbb{P}^2 \). Recall that given a quadric-line arrangement in \( \mathbb{P}^2 \), we are interested in computing the fundamental group of its complement. In [3], general presentations for some families of quadric-line arrangement...
were computed. The present paper is devoted to the computation of the fundamental groups related to three infinite families of quadric arrangements. The three types of interesting families of quadric curves in this paper are as follows. The first arrangement is a union of \( n \) quadrics, which are tangent to each other at two common points (figure 2). The second arrangement is composed of \( n \) quadrics which are tangent to each other at one common point (figure 3). The third arrangement is composed again of \( n \) quadrics, \( n - 1 \) of them are tangent to the \( n \)th one and each one of the \( n - 1 \) quadrics is transversal to the other \( n - 2 \) ones (figure 4).

Some work has been done concerning line arrangements (see e.g. [7,11,12]), and other progress has been done also concerning quadric-line arrangements (see [1–4]).

Let \( C \subset \mathbb{P}^2 \) be a plane curve and \( * \in \mathbb{P}^2 \setminus C \) a base point. By abuse of notation, we will call the group \( \pi_1(\mathbb{P}^2 \setminus C, *) \) the \textit{fundamental group} of \( C \), and shall frequently omit base points and write \( \pi_1(\mathbb{P}^2 \setminus C) \). One is interested in the group \( \pi_1(\mathbb{P}^2 \setminus C) \) mainly for two reasons. First, when the curve appears to be a branch curve, then \( \pi_1(\mathbb{P}^2 \setminus C) \) is an important invariant, concerning either the branch curve or the surface itself. Secondly, it contributes to the study of the Galois coverings \( X \to \mathbb{P}^2 \) branched along \( C \). Many interesting surfaces have been constructed as branched Galois coverings of the plane. An example has been already given in [3]. It involves the arrangement \( A_3 \) (shown in figure 1 above), which has Galois coverings \( X \to \mathbb{P}^2 \) branched along it, \( X \simeq \mathbb{P}^1 \times \mathbb{P}^1 \), or \( X \) is either an abelian surface, a \( K3 \) surface, or a quotient of the two-ball \( B_3 \) (see [8,14,16]). Moreover, some line arrangements defined by unitary reflection groups studied in [10] are related to \( A_3 \) via orbifold coverings. For example, if \( \mathcal{L} \) is the line arrangement given by the equation

\[
xyz(x + y + z)(x + y - z)(x - y + z)(x - y - z) = 0,
\]

then the image of \( \mathcal{L} \) under the branched covering map

\[
[x : y : z] \in \mathbb{P}^2 \to [x^2 : y^2 : z^2] \in \mathbb{P}^2
\]

is the arrangement \( A_3 \), see [14] for details.
The standard tool for fundamental group computations is the Zariski-van Kampen algorithm [15,17] (see [5] for a modern approach and [3] for a detailed explanation). We use a variation of this algorithm developed in [13] for computing the fundamental groups of quadrics arrangements and avoid lengthy monodromy computations. This approach has the advantage that it permits to capture the local fundamental groups around special and complicated singularities of these arrangements. The local fundamental groups are needed for the study of the singularities of $\mathbb{P}^2$ branched along these arrangements.

This paper is divided into four parts. In section 1, we quote basic definitions and an alternative way for computing quadric arrangements, by defining them as birational to line arrangements or as their coverings. In section 2, we quote the results of this work (Theorems 2.2, 2.5, 2.8), using the techniques from [3] and [13]. In sections 3, 4, and 5, we prove these theorems.

1. Quadric arrangements related to line arrangements

1.1. Meridians

Let $C \subset X$ be a curve in a smooth complex surface $X$ and $p \in C$. A meridian $\mu$ of $C$ at $p$, based on a point $* \in \mathbb{P}^2 \setminus C$ is a loop in $\mathbb{P}^2 \setminus C$ obtained by following a path $\omega$ with $\omega(0) = *$ and $\omega(1)$ belonging to a small neighborhood of $p$, turning around $C$ in the positive sense along the boundary of a small disc $\Delta$, having with $C$ a single intersection at $p$, and then turning back to $*$ along $\omega$. If $B \subset C$ is an irreducible component, a meridian $\mu_p$ of $C$ at a point $p \in B \setminus \text{Sing}(C)$ will be called a meridian of $B$. It is well-known that (homotopy classes of) any two meridians of $B$ are conjugate elements in $\pi_1(X \setminus C, *)$ (see e.g. [9, section 7.5]). When $p$ is a singular point of $C$, we have the following result.

**Lemma 1.1.** Let $p \in C$ be a singular point, $\mu_p$ a meridian of $C$ at $p$, and let $\sigma : Y \to \mathbb{P}^2$ be the blow-up of $X$ at $p$. Denote by $C$ the proper transform of $C$ and by $P$ the exceptional divisor. Then $\sigma(\mu_p)$ is a meridian of $P$. In particular, any two meridians of $C$ at $p$ are conjugate elements of $\pi_1(X \setminus C) \simeq \pi_1(Y \setminus (C \cup P))$.

**Proof.** The spaces $Y \setminus (C \cup P)$ and $X \setminus C$ are homeomorphic. By definition, $\mu_p = \omega \cdot \partial \Delta \cdot \omega^{-1}$, where $\Delta$ is a disc, having an intersection with $C$ at $p$, implying that the disc $\sigma(\Delta)$ intersects $P$ transversally and away from $C$. In other words, the loop $\sigma(\mu)$ is a meridian of $P$. \qed

The group $\pi_1(X \setminus C)$ is an invariant of the pair $(\mathbb{P}^2, C)$. Since meridians are well-defined up to a conjugacy class, they can be considered as supplementary invariants of $\pi_1(X \setminus C)$. What follows is a description of how to capture the meridians at singular points of $C$, during the computation of the group $\pi_1(X \setminus C)$ by Zariski-van Kampen.

**Lemma 1.2.** Let $C \subset \mathbb{P}^2$ be a curve, $L_0$ a line in general position with respect to $C$ and let $* \in L_0 \setminus C$ be a base point. Let $p$ be a singular point of $C$. We assume that
Let $L_0$ passes through a sufficiently small ball $V$ around $p$, so that the disc $\Delta := L_0 \cap V$ meets all branches of $C$ meeting at $p$. Take a path $\omega$ in $L_0$ connecting $*$ to a boundary point $q$ of $\Delta$. Then $\mu_0 := \omega \cdot \partial \Delta \cdot \omega^{-1}$ is a homotopic to a meridian of $C$ at $p$.

Proof. Let $L_1$ be the line through $*$ and $p$. Consider the projection $\phi : \mathbb{P}^2 \to \mathbb{P}^1$ from the point $*$ and with $\phi(L_0) = a$, $\phi(L_1) = b$ and take a path $\gamma \subset \phi(V)$ with $\gamma(0) = a$ and $\gamma(1) = b$. Put $L_t$ for the fiber above $\gamma(t)$. Let $\sigma \subset \partial V$ be a lift of the path $\gamma$ with $\sigma(0) = q$. Assume that $\sigma_t$ is the loop $\sigma$ until $t$, i.e. with $\sigma_t(t) := \sigma(\tau t) \ (t \in [0,1])$. Put $\omega_t := \sigma_t \cdot \omega$ and define $\mu_t := \omega_t \cdot \partial \Delta_t \cdot \omega_t^{-1}$, where $\Delta_t := L_t \cap V$. Then $M(t, \tau) := \mu_t(t)$ gives a homotopy between $\mu_0$ and $\mu_1$, and this latter loop is obviously a meridian of $C$ at $p$. 

1.2. Quadric arrangements birational to line arrangements

Assume that $\mathcal{A}$ is a line arrangement, and let $\psi$ be the involution

$$\psi : [x : y : z] \in \mathbb{P}^2 \to [1/x : 1/y : 1/z] \in \mathbb{P}^2.$$ 

Suppose that the lines $X, Y, Z$ are respectively given by the equations $x = 0, y = 0,$ and $z = 0$. If $\mathcal{A}$ is in general position with respect to $X \cup Y \cup Z$, then $\psi(\mathcal{A})$ is an arrangement of smooth quadrics. In addition to those of $\mathcal{A}$, this arrangement has three more singular points where all the irreducible components of $\psi(\mathcal{A})$ meet transversally.

In this case, the group $\pi_1(\mathbb{P}^2 \setminus \psi(\mathcal{A}))$ can easily be found in terms of $\pi_1(\mathbb{P}^2 \setminus \mathcal{A})$ as follows: Assuming $\mathcal{A} = \cup_{i=1}^m L_i$, let

$$\pi_1(\mathbb{P}^2 \setminus \mathcal{A}) \simeq \langle \mu_1, \ldots, \mu_n \mid w_1 = \cdots = w_m = \mu_1 \cdots \mu_n = 1 \rangle \quad (1)$$

be a presentation obtained by an application of Zariski-van Kampen, where $\mu_i$ is a meridian of $L_i$. Put $\mathcal{A}' := \mathcal{A} \cup X \cup Y \cup Z$. Since $\mathcal{A}$ is in general position with respect to $X \cup Y \cup Z$, one has by [7]

$$\pi_1(\mathbb{P}^2 \setminus \mathcal{A}') \simeq \left\langle \mu_1, \ldots, \mu_n, \frac{[\mu_i, \sigma_j] = [\sigma_j, \sigma_k] = 1 \ (i \in [1,n], j, k \in [1,3])}{\sigma_1, \sigma_2, \sigma_3} \mid w_1 = \cdots = w_m = \mu_1 \cdots \mu_n \sigma_1 \sigma_2 \sigma_3 = 1 \right\rangle \quad (2)$$

where $\sigma_1, \sigma_2, \sigma_3$ are, respectively, meridians of $X, Y,$ and $Z$. Let $p := X \cap Y$, $q := Y \cap Z$, and $r := Z \cap X$. Then $\sigma_1 \sigma_2$ (respectively, $\sigma_2 \sigma_3, \sigma_3 \sigma_1$) is a meridian of $\mathcal{A}'$ at $p$ (respectively, $q, r$). Hence, the group $\pi_1(\mathbb{P}^2 \setminus \psi(\mathcal{A}))$ can be obtained by setting $\sigma_1 \sigma_2 = \sigma_2 \sigma_3 = \sigma_3 \sigma_1 = 1$ in the presentation of $\pi_1(\mathbb{P}^2 \setminus \mathcal{A})$. But these relations imply $\sigma := \sigma_1 = \sigma_2 = \sigma_3$ and $\sigma^2 = 1$ and since by the projective relation one has $\mu_1 \cdots \mu_n \sigma_1 \sigma_2 \sigma_3 = 1$, it suffices to replace this latter relation by $(\mu_1 \cdots \mu_n)^2 = 1$. Hence

$$\pi_1(\mathbb{P}^2 \setminus \psi(\mathcal{A})) \simeq \left\langle \mu_1, \ldots, \mu_n \mid [\mu_i, \mu_1 \cdots \mu_n] = 1 \ (i \in [1,n]), w_1 = \cdots = w_m = (\mu_1 \cdots \mu_n)^2 = 1 \right\rangle.$$ 

Since $\sigma$ is a central element of this group, this proves the following result.
Theorem 1.3. For any arrangement of $n$ lines $\mathcal{A}$, there is an arrangement of $n$ smooth quadrics $\mathcal{B}$ with a central extension
\[ 0 \to \mathbb{Z}/(2) \to \pi_1(\mathbb{P}^2 \setminus \mathcal{B}) \to \pi_1(\mathbb{P}^2 \setminus \mathcal{A}) \to 0. \]

1.3. Quadric arrangements as coverings of line arrangements

Assume that $\mathcal{A}$ is a line arrangement, and let $\phi$ be the branched covering $\phi : [x : y : z] \in \mathbb{P}^2 \to [x^2 : y^2 : z^2] \in \mathbb{P}^2$.

Suppose that the lines $X, Y, Z$, are respectively given by the equations $x = 0, y = 0$, and $z = 0$. If $\mathcal{A}$ is in general position to $X \cup Y \cup Z$, then $\phi^{-1}(\mathcal{A})$ is an arrangement of smooth quadrics. Above any singular point of $\mathcal{A}$ lie four singular points of $\phi^{-1}(\mathcal{A})$ of the same type. In this case, the group $\pi_1(\mathbb{P}^2 \setminus \phi^{-1}(\mathcal{A}))$ can easily be found in terms of $\pi_1(\mathbb{P}^2 \setminus \mathcal{A})$ as follows: Assuming $\mathcal{A} = \bigcup_{i=1}^n L_i$, one has a presentation (1). For the arrangement $\mathcal{A}' := \mathcal{A} \cup X \cup Y \cup Z$, the presentation (2) is valid. There is an exact sequence
\[ 0 \to \pi_1(\mathbb{P}^2 \setminus \phi^{-1}(\mathcal{A}')) \to \pi_1(\mathbb{P}^2 \setminus \mathcal{A}') \to \mathbb{Z}/(2) \oplus \mathbb{Z}/(2) \to 0. \]

The group $\pi_1(\mathbb{P}^2 \setminus \phi^{-1}(\mathcal{A}))$ is the quotient $\pi_1(\mathbb{P}^2 \setminus \phi^{-1}(\mathcal{A}'))$ by the subgroup generated by the meridians of $\phi^{-1}(X), \phi^{-1}(Y)$ and $\phi^{-1}(Z)$.

2. Statements of results

In this section we give in Theorems 2.2, 2.5, and 2.8 the presentations of the fundamental groups of the three quadric arrangements in $\mathbb{P}^2$. We prove them in the forthcoming sections. For our computations, we need the following definition.

Definition 2.1. A group $G$ is said to be big if it contains a non-abelian free subgroup, and small if $G$ is almost solvable.

2.1. The quadric arrangement $\mathcal{A}_n$

Let $\mathcal{A}_n := Q_1 \cup \cdots \cup Q_n$ be a quadric arrangement, which is a union of $n$ quadrics tangent to each other at two common points, see figure 2.

Theorem 2.2. The fundamental group $\pi_1(\mathbb{P}^2 \setminus \mathcal{A}_n)$ of $\mathcal{A}_n$ in $\mathbb{P}^2$ admits the presentation
\[ \pi_1(\mathbb{P}^2 \setminus \mathcal{A}_n) \cong \langle a_1, a_2, \ldots, a_n \mid (a_n \cdots a_2 a_1)^2 = (a_1 a_n \cdots a_2)^2 = \cdots = (a_{n-1} a_{n-2} \cdots a_n)^2, a_n a_{n-1} \cdots a_2 a_1 a_2 \cdots a_{n-1} a_n = e \rangle, \quad (3) \]
where $a_1, \ldots, a_n$ are meridians of $Q_1, \ldots, Q_n$, respectively.
Proposition 2.3.

\[
\pi_1(\mathbb{P}^2 \setminus A_1) = \mathbb{Z}_2, \quad (4) \\
\pi_1(\mathbb{P}^2 \setminus A_2) \cong \left\langle a, b \mid (ab)^2 = (ba)^2, b^2a^2 = 1 \right\rangle \text{ is infinite and solvable}, \quad (5) \\
\pi_1(\mathbb{P}^2 \setminus A_3) \cong \left\langle a, b, c \mid (cba)^2 = (acb)^2 = (bac)^2, cba^2bc = 1 \right\rangle \text{ is big}. \quad (6)
\]

Corollary 2.4. The group \(\pi_1(\mathbb{P}^2 \setminus A_n)\) is big for \(n \geq 3\).

2.2. The quadric arrangement \(B_n\)

Let \(B_n := Q_1 \cup \cdots \cup Q_n\) be a quadric arrangement, composed of \(n\) quadrics tangent to each other at one common point, see Figure 3.

Theorem 2.5. The group \(\pi_1(\mathbb{P}^2 \setminus B_n)\) admits the presentation

\[
\pi_1(\mathbb{P}^2 \setminus B_n) \\
\cong \left\langle a_1, a_2, \ldots, a_n \mid (a_1 \cdots a_n)^4 = (a_na_1 \cdots a_{n-1})^4 = \cdots = (a_2 \cdots a_na_1)^4, a_n^2a_1^2 = 1 \right\rangle, \quad (7)
\]

where \(a_1, \ldots, a_n\) are meridians of \(Q_1, \ldots, Q_n\), respectively.
Proposition 2.6.

\[ \pi_1(\mathbb{P}^2 \setminus B_1) = \mathbb{Z}_2, \]
\[ \pi_1(\mathbb{P}^2 \setminus B_2) \cong \left\langle a, b \mid (ab)^4 = (ba)^4 \right\rangle \ 	ext{is infinite and solvable}, \]
\[ \pi_1(\mathbb{P}^2 \setminus B_3) \cong \left\langle a, b, c \mid (cba)^4 = (acb)^4 = (bac)^4 \right\rangle \ 	ext{is big}. \]

Corollary 2.7. The group \( \pi_1(\mathbb{P}^2 \setminus B_n) \) is big for \( n \geq 3 \).

2.3. The quadric arrangement \( C_n \)

Let \( C_n := Q_1 \cup \cdots \cup Q_n \) be a quadric arrangement, which is a union of \( n \) quadrics, \( n - 1 \) of them are tangent to the \( n \)th one and each one of these \( n - 1 \) quadrics is transversal to the other \( n - 2 \), see figure 4.

Theorem 2.8. The group \( \pi_1(\mathbb{P}^2 \setminus C_n) \) admits the presentation

\[ \pi_1(\mathbb{P}^2 \setminus C_n) \cong \left\langle a_1, \ldots, a_n \mid [a_i, a_j] = 1, \quad 2 \leq i, j \leq n, \quad i \neq j, \right. \]
\[ \left. (a_1a_k)^4 = (a_ka_1)^4, \quad 2 \leq k \leq n \right\rangle, \]
\[ a_2^n \cdots a_1^2 = 1 \]

where \( a_1, \ldots, a_n \) are meridians of \( Q_1, \ldots, Q_n \), respectively.
Proposition 2.9.

\[ \pi_1(P^2 \setminus C_1) = \mathbb{Z}_2, \quad \text{(12)} \]

\[ \pi_1(P^2 \setminus C_2) \simeq \left\langle a_1, a_2 \middle| (a_1a_2)^4 = (a_2a_1)^4, a_2^2a_1^2 = 1 \right\rangle \text{ is infinite and solvable,} \quad \text{(13)} \]

\[ \pi_1(P^2 \setminus C_3) \simeq \left\langle a_1, a_2, a_3 \middle| \begin{array}{l}
[a_2, a_3] = 1 \\
(a_1a_2)^4 = (a_2a_1)^4 \\
(a_1a_3)^4 = (a_3a_1)^4 \\
a_3^2a_2^2a_1^2 = 1
\end{array} \right\rangle. \quad \text{(14)} \]

3. Proof of Theorem 2.2

Take the following affine quadric arrangement \( \mathcal{A}_n \), which is composed of \( n \) quadrics tangent to each other at two points, see figure 2. Say that these points are \((1,0)\) and \((-1,0)\). Let \( p_1 : \mathbb{C}^2 \setminus \mathcal{A}_n \to \mathbb{C} \) be the first projection. The base of this projection will be denoted by \( B \). Identify the base \( B \) of the projection \( p_1 \) with the line \( y = -2 \subset \mathbb{C}^2 \).

The projections of \((1,0)\) and \((-1,0)\) are \((1,-2)\) and \((-1,-2)\), respectively. Take \( \ast := (M,-2) \) to be the base point. Put \( F_x := p_1^{-1}(x) \), and denote by \( S \) the set of singular fibers of \( p_1 \). It is clear that if \( F_x \in S \), then \( x \in [-1,1] \). Therefore, the only singular fibers are \( F_1 \) and \( F_{-1} \), both corresponding to \((1,0)\) and \((-1,0)\).

In order to compute the fundamental group \( \pi_1(\mathbb{P}^2 \setminus \mathcal{A}_n) \), we have to study first the
local monodromy around the points $(1,0)$ and $(-1,0)$. This is done in the following proposition.

**Proposition 3.1.** Let $0$ be a point (figure 5) which is a tangency of $n$ quadrics defined locally by

$$(y - x^2)(y - 2x^2)(y - 3x^2) \cdots (y - nx^2).$$

Then the local monodromy around $0$ is a full twist $H^2$ of the $n$ points ($H$ is a counterclockwise half twist).

**Proof.** Take a loop $x = e^{2\pi it}$ in $y = 0$, starting (and ending) at the point $*$ and encircling the point $0$, $0 \leq t \leq 1$. Take a typical fiber $F_*$ next to the fiber $F_0$. We have $n$ points on $F_*$, say $1,2,\ldots,n$ (the intersection points of the arrangement with the fiber).

When $t = 0$, the points $1,2,\ldots,n$ are still in their initial positions. When $t$ is proceeding from 0 to 1, there is an induced motion of the points. The point 1 is rotating around the other points with a full twist $e^{2\pi i}$, the point 2 is rotating along a closed curve which bounds a disk, containing the trajectory of the point 1, and so on. This gives the required monodromy.

**Proof of Theorem 2.2.** Let us denote the loops around the points $1,\ldots,n$ in Proposition 3.1 as $a_1,\ldots,a_n$, respectively. These loops get the forms $\tilde{a}_1,\ldots,\tilde{a}_n$, as depicted in figure 6.

By the Zariski Theorem [17],

$$\tilde{a}_1 = a_n \cdots a_2 a_1 a_n^{-1} a_2^{-1} \cdots a_n^{-1} a_1^{-1} a_2^{-1} \cdots a_n^{-1} a_1^{-1} \cdots a_n^{-1} a_1^{-1} \cdots a_n^{-1}.$$
These relations get the forms

\[(a_n \cdots a_1)^2 = (a_1 a_n \cdots a_2)^2\]
\[(a_n \cdots a_1)^2 = (a_2 a_1 a_n \cdots a_3)^2\]
\[\vdots\]
\[(a_n \cdots a_1)^2 = (a_{n-1} a_{n-2} \cdots a_1 a_n)^2.\]

Since the points \((1, 0)\) and \((-1, 0)\) are intersections of branch points, we have

\[a_i = a'_i \quad \text{for } 1 = 1, \ldots, n.\]

The projective relation

\[a'_1 a'_2 \cdots a'_n a_n \cdots a_2 a_1 = 1\]

gets the form

\[a_1 a_2 \cdots a_n^2 \cdots a_2 a_1 = 1.\]

Therefore, the group \(\pi_1(\mathbb{P}^2 \setminus A_n)\) admits (3). \(\square\)

**Proof of Proposition 2.3.** It is easy to see that the arrangement \(A_1\) consists of a smooth quadric, and the group is \(\mathbb{Z}_2\), see (4).

For the case \(n = 2\), we depict figure 7. We substitute \(n = 2\) in (3) and get the presentation (5). Now we prove that \(\pi_1(\mathbb{P}^2 \setminus A_2)\) is an infinite solvable group. Apply
a change of generators in (5): \( \alpha := ba, \beta := b \), and get

\[
\pi_1(\mathbb{P}^2 \setminus A_2) \cong \left\langle \alpha, \beta \left| \begin{array}{c}
\beta \alpha^2 \beta^{-1} = \alpha^2 \\
\alpha^{-1} \beta = \beta \alpha
\end{array} \right. \right\rangle.
\]

The first relation gets the form (by the second one) \( \alpha^{-1} \beta \alpha \beta^{-1} = \alpha^2 \). This yields \( \beta \alpha \beta^{-1} = \alpha^3 \). We repeat this procedure and obtain \( \alpha^4 = 1 \). Therefore the subgroup generated by \( \alpha \) is a finite one. Since \( \beta \alpha \beta^{-1} = \alpha^3 \), \( \beta \alpha^3 \beta^{-1} = \alpha \), and \( \beta \alpha^2 \beta^{-1} = \alpha^2 \), this subgroup is normal. Sending \( \alpha \to 1 \), the image is generated by \( \beta \) and no relations remain. Thus \( \pi_1(\mathbb{P}^2 \setminus A_2) \) is an extension of \( \mathbb{Z}_4 \) by \( \mathbb{Z} \), and in particular an infinite solvable one.

Now, when we substitute \( n = 3 \) in (3) we get (6). We prove that \( \pi_1(\mathbb{P}^2 \setminus A_3) \) is big. We apply a change of generators in (6), \( \alpha := cb \alpha, \beta := cb, \gamma := b \), which gives

\[
\pi_1(\mathbb{P}^2 \setminus A_3) \cong \left\langle \alpha, \beta, \gamma \left| \begin{array}{c}
[\alpha^2, \beta] = [\alpha^2, \gamma] = 1 \\
\gamma \beta \gamma^{-1} = \alpha^{-1} \beta \alpha^{-1}
\end{array} \right. \right\rangle.
\]

There is a surjection of \( \pi_1(\mathbb{P}^2 \setminus A_3) \) onto its quotient, by adding \( \alpha^2 = 1 \)

\[
\left\langle \alpha, \beta, \gamma \left| \begin{array}{c}
\alpha^2 = 1 \\
\alpha \gamma \beta = \beta \alpha \gamma
\end{array} \right. \right\rangle.
\]  (15)
An isomorphism $\alpha, \beta, \gamma \to \alpha, \beta, \alpha^{-1}\gamma$, respectively, gives

\[ \langle \alpha, \beta, \gamma \mid \alpha^2 = 1, \gamma\beta = \beta\gamma \rangle. \]

Mapping $\beta \to 1$ and fixing $\gamma^3 = 1$, we get $\mathbb{Z}_2 * \mathbb{Z}_3$. This group has a free subgroup [6], therefore $\pi_1(\mathbb{P}^2 \setminus \mathcal{A}_3)$ is big.

**Proof of Corollary 2.4.** $\pi_1(\mathbb{P}^2 \setminus \mathcal{A}_3)$ is a quotient group of $\pi_1(\mathbb{P}^2 \setminus \mathcal{A}_n)$ ($n \geq 3$), and $\pi_1(\mathbb{P}^2 \setminus \mathcal{A}_3)$ is big, therefore $\pi_1(\mathbb{P}^2 \setminus \mathcal{A}_n)$ is a big group.

### 4. Proof of Theorem 2.5

The arrangement $\mathcal{B}_n$ is defined by

\[ (x^2 + y^2 - 1) \bigg( \left( x - \frac{1}{2} \right)^2 + y^2 - \frac{1}{4} \bigg) \bigg( \left( x - \frac{3}{4} \right)^2 + y^2 - \frac{1}{16} \bigg) \cdots. \]

The quadrics $Q_1, \ldots, Q_n$ are tangent to each other at one common tangency $(1, 0)$, see figure 3. As in the case of the arrangement $\mathcal{A}_n$, it is readily seen that arrangements of type $\mathcal{B}_n$ are all isotopic to each other for fixed $n$.

The projection to the line $y = -2$ has two types of singular fibers:

(i) the fibers $F_{-1}, F_0, F_1, \ldots$ corresponding to branch points of the $n$ quadrics,

(ii) the fiber $F_1$ corresponding to the tangency $(1, 0)$.

In order to find the group $\pi_1(\mathbb{P}^2 \setminus \mathcal{B}_n)$, we shall apply the same procedure as in section 3. Let $*$ be the base point and $F_*$ a typical fiber. Denote the branch points of the quadrics as $s_1, \ldots, s_n$ and $(1, 0)$ as $s_{n+1}$. Say that the typical fiber intersects $Q_1$ at $a_1, a'_1$, $Q_2$ at $a_2, a'_2$, and so on.

First we study the local monodromy around the point $s_{n+1}$.

**Proposition 4.1.** Define the unique tangency of $n$ quadrics (e.g. figure 5) locally by

\[ (y - x^4)(y - 2x^4)(y - 3x^4) \cdots (y - nx^4). \]

Let $a_1, \ldots, a_n$ be the intersection points of the quadrics with a typical fiber. Then the local monodromy around this tangency is a double fulltwist $H^4$ of the points $a_1, \ldots, a_n$.

**Proof.** Take a loop $x = e^{2\pi i t}$ in $y = 0$, starting (and ending) at the point $*$ and encircling the point $0, 0 \leq t \leq 1$.

When $t = \frac{1}{2}$, the resulting motion of the points is: the point $a_1$ is turning around all the points in a twist of $e^{4\pi i}$, the point $a_2$ is turning around in a bigger twist of $2e^{3\pi i}$, and so on.
Proof of Theorem 2.5. We compute the group \( \pi_1(\mathbb{P}^2 \setminus \mathcal{B}_n) \). Take a \( g \)-base \( a_1, \ldots, a_n \) for \( F_+ \setminus \{a_1, \ldots, a_n\} \).

By Proposition 4.1, for \( t = 1 \) the motion of the \( n \) points is a \( e^{8\pi i} \) twist. The branch points \( s_1, \ldots, s_n \) contribute \( a_i = a'_i \) for \( i = 1, \ldots, n \), and together with the projective relation, lengthy computations yield the presentation (7).

Proof of Proposition 2.6. It is easy to see that the arrangement \( \mathcal{B}_1 \) consists of a smooth quadric, and the group is \( \mathbb{Z}_2 \), see (8).

The arrangement \( \mathcal{B}_2 \) is shown in figure 8. Substituting \( n = 2 \) in (7) yields (9).

Now we show another way to find \( \pi_1(\mathbb{P}^2 \setminus \mathcal{B}_2) \). We understand first the local monodromy around the point \((1,0)\). A tangency of two quadrics is homotopic equivalent to a tangency of a line with a quadric. Therefore, we can use figure 9 as a local model defined by \( y(y-x^4) = 0 \).

Lemma 4.2. The local monodromy around the point 0 is a twist \( H^4 \) of \( b \) around \( a \).

Proof. Since we explained already the construction of a \( g \)-base for a fundamental group in [3], we can directly compute the local monodromy around the point 0. Take
a loop \( x = e^{2\pi it} \) in \( y = 0 \), starting (and ending) at the point \(*\) and encircling the point \( 0, 0 \leq t \leq 1 \). Take, for example, the fiber \( F_1 \) in figure 9. The points on this fiber are \( a = (1,0) \) and \( b = (1,1) \). They are encircled by the loops (liftings of \( x = e^{2\pi it} \)) \( a \) and \( b \) of \( \pi_1(\mathbb{P}^2 \setminus \mathcal{B}_2) \). It is easy to see that when \( t = \frac{1}{2} \), the induced motion is a double full twist of \( b \) around \( a \).

When \( t = 1 \), the point \( y = 0 \) is left in its place, while the second point is now \( y = e^{8\pi i} \). That means that the second point was encircling 0 in four counterclockwise full twists. The resulting loops \( \tilde{a} \) and \( \tilde{b} \) are shown in figure 10 and formulated as follows:

\[
\begin{align*}
\tilde{a} &= bababab^{-1}a^{-1}b^{-1}a^{-1}b^{-1}a^{-1}b^{-1} \\
\tilde{b} &= babababa^{-1}b^{-1}a^{-1}b^{-1}a^{-1}b^{-1}a^{-1}b^{-1}.
\end{align*}
\]

These relations get the form

\[(ab)^4 = (ba)^4.\]

\( s_1 \) and \( s_2 \) are branch points (figure 8), therefore \( a_1 = b_1 \) and \( a_2 = b_2 \). The projective relation

\[b_1b_2a_2a_1 = 1\]

is transformed to

\[b^2a^2 = 1.\]

Therefore we obtain (9).

We prove that \( \pi_1(\mathbb{P}^2 \setminus \mathcal{B}_2) \) is an infinite solvable group. Apply a change of generators in (9), \( \alpha := ba, \beta := b \), and get

\[
\pi_1(\mathbb{P}^2 \setminus \mathcal{B}_2) \simeq \left\langle \alpha, \beta \mid \beta \alpha^4 \beta^{-1} = \alpha^4, \beta \alpha \beta^{-1} = \alpha^{-1} \right\rangle.
\]
Figure 10: Resulting generators $\tilde{a}, \tilde{b}$ in $\pi_1(\mathbb{P}^2 \setminus B_2)$
The first relation is transformed by the second relation to $\alpha^{-1}\beta\alpha^3\beta^{-1} = \alpha^4$. This yields $\beta\alpha^3\beta^{-1} = \alpha^5$. We repeat this procedure and obtain $\alpha^8 = 1$. Therefore the subgroup generated by $\alpha$ is a finite one. One can prove that it is also normal. Mapping $\alpha \to 1$, the image is generated by $\beta$ and no relations remain. Thus $\pi_1(\mathbb{P}^2 \setminus \mathcal{B}_2)$ is an extension of $\mathbb{Z}_8$ by $\mathbb{Z}$, and in particular infinite and solvable.

Now, substituting $n = 3$ in (7), we obtain (10). We prove that it is a big group. By a change of generators $\kappa := cba$, $\tau := cb$, $m := c$ in (10), we obtain

$$\pi_1(\mathbb{P}^2 \setminus \mathcal{B}_3) \simeq \langle \kappa, \tau, m \mid [\kappa^4, \tau] = [\kappa^4, m] = 1 \rangle.$$ 

Adding the relation $\kappa^2 = 1$, we have a surjection of $\pi_1(\mathbb{P}^2 \setminus \mathcal{B}_3)$ onto

$$\langle \kappa, \tau, m \mid [\kappa m, \tau] = 1 \rangle,$$

which is equal to (15). At this point, we have to repeat the proof which we did for $\pi_1(\mathbb{P}^2 \setminus \mathcal{A}_3)$, and then we get $\pi_1(\mathbb{P}^2 \setminus \mathcal{B}_3)$ is also big.

**Proof of Corollary 2.7.** The group $\pi_1(\mathbb{P}^2 \setminus \mathcal{B}_3)$ is a quotient of $\pi_1(\mathbb{P}^2 \setminus \mathcal{B}_n)$ ($n \geq 3$), and $\pi_1(\mathbb{P}^2 \setminus \mathcal{B}_3)$ is big, therefore $\pi_1(\mathbb{P}^2 \setminus \mathcal{B}_n)$ is big.

5. **Proof of Theorem 2.8**

Let $\mathcal{C}_n$ be a quadric arrangement, composed of $n$ quadrics as shown in Figure 4. Each one of the quadrics $Q_2, \ldots, Q_n$ is tangent to the quadric $Q_1$ in one tangency and intersects each one of the other quadrics at two points.

The projection to the line $y = -2$ has three types of singular fibers:

(i) the fibers $F_{s_i}$, where $s_i$ is a branch point of a quadric $Q_i$, $1 \leq i \leq n$,

(ii) the fibers $F_{x_i}$ corresponding to the nodes,

(iii) the fibers $F_{t_i}$ corresponding to the $n - 1$ tangencies.

In order to find the group $\pi_1(\mathbb{P}^2 \setminus \mathcal{C}_n)$, take $* \in \mathbb{R}$ a base point and $F_*$ a typical fiber. Next to each branch point one can find a fiber which intersects the same quadric in two real points. Denote this pair of points as $a_i$, $b_i$ for $1 \leq i \leq n$.

**Proof of Theorem 2.8.** Since we are familiar with the types of the singularities, we can compute easily the group $\pi_1(\mathbb{P}^2 \setminus \mathcal{C}_n)$. The nodes and tangencies give $[a_i, a_j] = 1$ for $i \neq j$, $2 \leq i, j \leq n$, and $(a_2a_k)^4 = (a_k^{-1}a_1)^4$ for $2 \leq k \leq n$. The branch points $s_1, \ldots, s_n$ give $a_i = b_i$. The projective relation is $a_2 \cdots a_1^2 = 1$. Therefore we get 2.8. \hfill \Box
Proof of Proposition 2.9. The group $\pi_1(\mathbb{P}^2 \setminus C_1)$ is again $\mathbb{Z}_2$, see (12).

The group $\pi_1(\mathbb{P}^2 \setminus C_2)$ admits the presentation (13), hence is isomorphic to the group $\pi_1(\mathbb{P}^2 \setminus B_2)$. Therefore it is an infinite solvable group.

The group $\pi_1(\mathbb{P}^2 \setminus C_3)$ admits (14). We apply a change of generators in (14), $\kappa := a_2a_1$, $\tau := a_2$, $m := a_3$, which yields a better look at the group:

$$\pi_1(\mathbb{P}^2 \setminus C_3) \cong \left\langle \kappa, \tau, m \mid [m, \tau] = 1, [\kappa^4, \tau] = 1, (\tau^{-1}km)^4 = (m\tau^{-1}\kappa)^4, \tau\kappa^{-1}km^2 = 1 \right\rangle. \quad \square$$

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References


