Global Gronwall Estimates for Integral Curves on Riemannian Manifolds

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ABSTRACT

We prove Gronwall-type estimates for the distance of integral curves of smooth vector fields on a Riemannian manifold. Such estimates are of central importance for all methods of solving ODEs in a verified way, i.e., with full control of roundoff errors. Our results may therefore be seen as a prerequisite for the generalization of such methods to the setting of Riemannian manifolds.

Key words: Riemannian geometry, ordinary differential equations, Gronwall estimate.


Introduction

Suppose that $X$ is a complete smooth vector field on $\mathbb{R}^n$, let $p_0, q_0 \in \mathbb{R}^n$ and denote by $p(t), q(t)$ the integral curves of $X$ with initial values $p_0$ resp. $q_0$. In the theory of

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ordinary differential equations it is a well known consequence of Gronwall’s inequality that in this situation we have
\[ |p(t) - q(t)| \leq |p_0 - q_0| e^{C_T t} \quad (t \in [0, T]) \] (1)
with \( C_T = \| DX \|_{L^\infty(K_T)} \) \((K_T\) some compact convex set containing the integral curves \( t \mapsto p(t) \) and \( t \mapsto q(t) \)) and \( DX \) the Jacobian of \( X \) (cf., e.g., [1, 10.5]).

The aim of this paper is to derive estimates analogous to (1) for integral curves of vector fields on Riemannian manifolds. Apart from a purely analytical interest in this generalization, we note that Gronwall-type estimates play an essential role in the convergence analysis of numerical methods for solving ordinary differential equations (cf., [5]). Concerning notation and terminology from Riemannian geometry our basic references are [2–4].

1. Estimates

The following proposition provides the main technical ingredient for the proofs of our Gronwall estimates. Here and in what follows, for \( X \in \mathfrak{X}(M) \) (the space of smooth vector fields on \( M \)) we denote by \( \nabla X \) its covariant differential and by \( \| \nabla X(p) \|_g \) the mapping norm of \( \nabla X(p) : (T_p M, \| \cdot \|_g) \rightarrow (T_p M, \| \cdot \|_g) \), \( Y_p \mapsto \nabla_{Y_p} X \).

**Proposition 1.1.** Let \([a, b] \ni \tau \mapsto c_0(\tau) = c(0, \tau)\) be a smooth regular curve in a Riemannian manifold \((M, g)\), let \( X \in \mathfrak{X}(M) \) and set \( c(t, \tau) := F_{l_0}^X c(0, \tau) \) where \( F_{l_0}^X \) is the flow of \( X \). Choose \( T > 0 \) such that \( F_{l_0}^X \) is defined on \([0, T] \times c_0([a, b])\). Then denoting by \( l(t) \) the length of \( \tau \mapsto c(t, \tau) \), we have
\[ l(t) \leq l(0) e^{C_T t} \quad (t \in [0, T]) \] (2)
where \( C_T = \sup \{ \| \nabla X(p) \|_g : p \in c([0, T] \times [a, b]) \} \).

**Proof.** Let \( \tau \mapsto c(0, \tau) \) be parameterized by arc length, \( \tau \in [0, l(0)] \). Since \( F_{l_0}^X \) is a local diffeomorphism, \( g(\partial_\tau c, \partial_\tau c) > 0 \) on \([0, T] \times [a, b] \). Furthermore, since the Levi-Civit\( \grave{a}\) connection \( \nabla \) is torsion free, we have \( \nabla_{\partial_\tau} c_\tau = \nabla_{\partial_\tau} c_\tau \), where \( c_\tau = \partial_\tau c, c_\tau = \partial_\tau c \), see [3, 1.8.14]. Then
\[
l(s) - l(0) = \int_0^s \partial_\tau l(t) \, dt = \int_0^s \partial_\tau \int_0^t \| c_\tau(t, \tau) \|_g \, d\tau \, dt
\]
\[
= \int_0^s \int_0^t \frac{\partial_\tau g(c_\tau(t, \tau), c_\tau(t, \tau))}{2\| c_\tau(t, \tau) \|_g} \, d\tau \, dt = \int_0^s \int_0^t \frac{g((\nabla_{\partial_\tau} c)(t, \tau), c_\tau(t, \tau))}{\| c_\tau(t, \tau) \|_g} \, d\tau \, dt
\]
\[
= \int_0^s \int_0^t \frac{g((\nabla_{\partial_\tau} c)(t, \tau), c_\tau(t, \tau))}{\| c_\tau(t, \tau) \|_g} \, d\tau \, dt \leq \int_0^s \int_0^t \| (\nabla_{\partial_\tau} c)(t, \tau) \|_g \, d\tau \, dt
\]

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The claim now follows by applying Gronwall’s inequality.

We may utilize this proposition to prove our first main result:

**Theorem 1.2.** Let \((M, g)\) be a connected smooth Riemannian manifold, \(X \in \mathfrak{X}(M)\) a complete vector field on \(M\) and let \(p_0, q_0 \in M\). Let \(p(t) = \text{Fl}^X_t(p_0), q(t) = \text{Fl}^X_t(q_0)\) and suppose that \(C := \sup_{p \in M} \|\nabla X(p)\|_g < \infty\). Then

\[
d(p(t), q(t)) \leq d(p_0, q_0)e^{Ct} \quad (t \in [0, \infty)),
\]

where \(d(p, q)\) denotes Riemannian distance.

**Proof.** For any given \(\varepsilon > 0\), choose a piecewise smooth regular curve \(\tau \mapsto c_\tau (0) = \cdots c_\tau (t) : [0, 1] \rightarrow M\) connecting \(p_0\) and \(q_0\) such that \(d(p_0, q_0) > l(0) - \varepsilon\). Using the notation of Proposition 1.1 it follows that

\[
d(p(t), q(t)) \leq l(t) \leq l(0)e^{Ct} < (d(p_0, q_0) + \varepsilon)e^{Ct}
\]

for \(t \in [0, \infty)\). Since \(\varepsilon > 0\) was arbitrary, the result follows.

**Example 1.3.**

(i) In general, when neither \(M\) nor \(X\) is complete, the conclusion of Theorem 1.2 is no longer valid:

Consider \(M = \mathbb{R}^2 \setminus \{(0, y) \mid y \geq 0\}\), endowed with the standard Euclidean metric. Let \(X \equiv (0, 1), p_0 = (-x_0, -y_0), \) and \(q_0 = (x_0, -y_0) (x_0 > 0, y_0 \geq 0)\) (cf. figure 1). Then \(p(t) = (-x_0, -y_0 + t), q(t) = (x_0, -y_0 + t)\) and

\[
d(p(t), q(t)) = \begin{cases} 2x_0, & t \leq y_0, \\ 2\sqrt{x_0^2 + (t - y_0)^2}, & t > y_0. \end{cases}
\]

On the other hand, \(\nabla X = 0\), so (3) is violated for \(t > y_0\), i.e., as soon as the two trajectories are separated by the “gap” \(\{(0, y) \mid y \geq 0\}\).

(ii) Replace \(X\) in (i) by the complete vector field \((0, e^{-1/\sqrt{2}^2}+1)\) and set \(x_0 = 1, y_0 = 0\). Then \(C := \|\nabla X\|_{L^\infty(\mathbb{R}^2)} = 3\sqrt{3}/(2\varepsilon)\) and

\[
d(p(t), q(t)) = 2\sqrt{1 + t^2} \leq d(p_0, q_0)e^{Ct} = 2e^{Ct}
\]

for all \(t \in [0, \infty)\), in accordance with Theorem 1.2.
The following result provides a sufficient condition for the validity of a Gronwall estimate even if neither $M$ nor $X$ satisfies a completeness assumption.

**Theorem 1.4.** Let $(M,g)$ be a connected smooth Riemannian manifold, $X \in \mathfrak{X}(M)$ and let $p_0, q_0 \in M$. Let $p(t) = \Phi_t^X(p_0)$, $q(t) = \Phi_t^X(q_0)$ and suppose that there exists some relatively compact submanifold $N$ of $M$ containing $p_0, q_0$ such that $d(p_0, q_0) = d_N(p_0, q_0)$. Fix $T > 0$ such that $\Phi_t^X$ is defined on $[0, T] \times N$ and set $C_T := \sup \{ \| \nabla X(p) \|_g : p \in \Phi_t^X([0, T] \times N) \}$. Then

$$d(p(t), q(t)) \leq d(p_0, q_0) e^{CT} \quad (t \in [0, T]).$$

**Proof.** As in the proof of Theorem 1.2, for any given $\varepsilon > 0$ we may choose a piecewise smooth curve $\tau \mapsto c_0(\tau) : [0, 1] \to N$ from $p_0$ to $q_0$ such that $d(p_0, q_0) = d_N(p_0, q_0) > l(0) - \varepsilon$. The corresponding time evolutions $c(t, \cdot)$ of $c(0, \cdot) = c_0$ then lie in $\Phi_t^X([0, T] \times N)$, so an application of Proposition 1.1 gives the result. \qed

**Example 1.5.** Clearly such a submanifold $N$ need not exist in general. As a simple example take $M = \mathbb{R}^2 \setminus \{(0, 0)\}$, $p_0 = (-1, 0)$, $q_0 = (1, 0)$. In Example 1.3.(i) with $y_0 > 0$ the condition is obviously satisfied with $N$ an open neighborhood of the straight line joining $p_0, q_0$ and the supremum of the maximal evolution times of such $N$ under $\Phi_t^X$ is $T = y_0$, coinciding with the maximal time-interval of validity of (4). On the other hand, if there is no $N$ as in Theorem 1.4 then the conclusion in general breaks down even for arbitrarily close initial points $p_0, q_0$: if we set $y_0 = 0$ in Example 1.3.(i) then no matter how small $x_0$ (i.e., irrespective of the initial distance of the trajectories) the estimate is not valid for any $T > 0$.

Finally, we single out some important special cases of Theorem 1.4:

**Corollary 1.6.** Let $M$ be a connected geodesically complete Riemannian manifold, $X \in \mathfrak{X}(M)$, and $p_0, q_0, p(t), q(t)$ as above. Let $S$ be a minimizing geodesic segment $S$.
connecting \( p_0, q_0 \) and choose some \( T > 0 \) such that \( F^X_t \) is defined on \([0,T] \times S\). Then (4) holds with \( C_T = \sup \{ \| \nabla X(p) \|_g \mid p \in F^X([0,T] \times S) \} \). In particular, if \( X \) is complete then for any \( T > 0 \) we have
\[
d(p(t), q(t)) \leq d(p_0, q_0) e^{C_T t} \quad (t \in [0,T]).
\]

Proof. Choose for \( N \) in Theorem 1.4 any relatively compact open neighborhood of \( S \). The value of \( C_T \) then follows by continuity.

In particular, for \( M = \mathbb{R}^n \) with the standard Euclidean metric, Corollary 1.6 reproduces (1).

References


