ON THE SINGULAR NUMBERS FOR SOME INTEGRAL OPERATORS

A. MESKHI

Abstract

Two–sided estimates of Schatten–von Neumann norms for weighted Volterra integral operators are established. Analogous problems for some potential-type operators defined on $\mathbb{R}^n$ are solved.

Let $H$ be a separable Hilbert space and let $\sigma_\infty(H)$ be the class of all compact operators $T : H \rightarrow H$, which forms an ideal in the normed algebra $B$ of all bounded linear operators on $H$. To construct a Schatten-von Neumann ideal $\sigma_p(H)$ ($0 < p \leq \infty$) in $\sigma_\infty(H)$, the sequence of singular numbers $s_j(T) \equiv \lambda_j(\|T\|)$ is used, where the eigenvalues $\lambda_j(\|T\|)$ ($\|T\| \equiv (T^*T)^{1/2}$) are non-negative and are repeated according to their multiplicity and arranged in decreasing order. A Schatten-von Neumann quasinorm (norm if $1 \leq p \leq \infty$) is defined as follows:

$$\|T\|_{\sigma_p(H)} \equiv \left( \sum_j s_j^p(T) \right)^{1/p}, \quad 0 < p < \infty,$$

with the usual modification if $p = \infty$. Thus we have $\|T\|_{\sigma_\infty(H)} = \|T\|$ and $\|T\|_{\sigma_2(H)}$ is the Hilbert-Schmidt norm given by the formula

$$\|T\|_{\sigma_2(H)} = \left( \int \int |T_1(x,y)|^2 \, dx \, dy \right)^{1/2} \quad (1)$$

for an integral operator

$$Tf(x) = \int T_1(x,y) f(y) \, dy.$$

We refer, for example, to [2], [6], [7] for more information concerning Schatten-von Neumann ideals.

2000 Mathematics Subject Classification: 26A33, 42B15, 47B10, 47G10.
Servicio de Publicaciones. Universidad Complutense. Madrid, 2001
In this paper necessary and sufficient conditions for the weighted Volterra integral operator

\[ K_v f(x) = v(x) \int_0^x f(y) k(x, y) dy, \quad x \in (0, a), \]

to belong to Schatten-von Neumann ideals are established, where \( v \) is a measurable function on \((0, a)\) (\( 0 < a \leq \infty \)).

Two-sided estimates of Schatten-von Neumann \( p \)-norms for the weighted Riemann–Liouville operator

\[ R_{\alpha, v} f(t) = v(x) \int_0^x f(t)(x - t)^{\alpha - 1} dt, \]

when \( \alpha > 1/2 \) and \( p > 1/\alpha \), were established in [13] (for \( \alpha = 1 \) and \( p > 1 \) see [14]). Analogous results for the weighted Hardy operator

\[ H_{v, u} f(x) = v(x) \int_0^x u(y) f(y) dy \]

were obtained in [3]. Similar problems for the Riemann-Liouville operator with two weights

\[ R_{\alpha, v, u} f(x) = v(x) \int_0^x u(t) f(t)(x - t)^{\alpha - 1} dt, \]

when \( \alpha \in \mathbb{N} \) and \( p \geq 1 \), were solved in [4]. Further, upper and lower bounds for Schatten–von Neumann \( p \)-norms (\( p \geq 2 \)) of certain Volterra integral operators, involving \( R_{\alpha, v, u} \) only for \( \alpha \geq 1 \), were proved in [4] and [18].

Our main goal is to generalize the results of [13] and [14] for integral transforms with kernels and to give two-sided estimates of the above-mentioned norms for these operators in terms of their kernels.

We denote by \( L^p_w(\Omega) \), \( \Omega \subseteq \mathbb{R}^n \), a weighted Lebesgue space with respect to the weight \( w \) defined on \( \Omega \).

Throughout the paper the expression \( A \approx B \) is interpreted as \( c_1 A \leq B \leq c_2 A \) with some positive constants \( c_1 \) and \( c_2 \).

Let us recall some definitions from [10] (see also [8]).

We say that a kernel \( k : \{(x, y) : 0 < y < x < a\} \rightarrow \mathbb{R}_+ \) belongs to \( V \) (\( k \in V \)) if there exists a positive constant \( d_1 \) such that for all \( x, y, z \) with \( 0 < y < z < x < a \) the inequality

\[ k(x, y) \leq d_1 k(x, z) \]
holds. Further, \( k \in V_\lambda (1 < \lambda < \infty) \) if there exists a positive constant \( d_2 \) such that for all \( x, x \in (0, a) \), the inequality
\[
\int_{x/2}^x k^\lambda(x, y)dy \leq d_2 x k^\lambda(x, x/2), \quad \lambda' = \frac{\lambda}{\lambda - 1}.
\]
is fulfilled.

For example, if \( k_1(x) = x^{\alpha - 1} \), where \( \frac{1}{\lambda} < \alpha \leq 1 \), then \( k(x, y) = k_1(x - y) \) belongs to \( V \cap V_\lambda \) (for other examples of kernel \( k \) see [10], [8]).

First we investigate the mapping properties of \( K_v \) in Lebesgue spaces. The following statements in equivalent form were proved in [10] (see also [8], [11]).

**Theorem A.** Let \( 1 < p \leq q < \infty \), \( a = \infty \) and let \( k \in V \cap V_p \). Then

(a) \( K_v \) is bounded from \( L^p(0, \infty) \) into \( L^q(0, \infty) \) if and only if
\[
D_\infty \equiv \sup_{j \in \mathbb{Z}} D_\infty(j) \equiv \sup_{j \in \mathbb{Z}} \left( \int_2^{2j+1} k^q(x, x/2) x^{q/p'} |v(x)|^q dx \right)^{1/q} < \infty.
\]
Moreover, \( \|K_v\| \approx D_\infty \).

(b) \( K_v \) acts compactly from \( L^p(0, a) \) into \( L^q(0, a) \) if and only if \( D_\infty < \infty \) and \( \lim_{j \to +\infty} D_\infty(j) = \lim_{j \to -\infty} D_\infty(j) = 0 \).

**Theorem B.** Let \( 1 < p \leq q < \infty \), \( a < \infty \) and let \( k \in V \cap V_p \). Then

(a) \( K_v \) is bounded from \( L^p(0, a) \) to \( L^q(0, a) \) if and only if
\[
D_a \equiv \sup_{j \geq 0} D_a(j) \equiv \sup_{j \geq 0} \left( \int_{2^{-j+1}a}^{2^{-j}a} |v(x)|^q k^q(x, x/2) x^{q/p'} dx \right)^{1/q} < \infty.
\]
Moreover, \( \|K_v\| \approx D_a \).

(b) \( K_v \) acts compactly from \( L^p(0, a) \) into \( L^q(0, a) \) if and only if \( D_a < \infty \) and \( \lim_{j \to +\infty} D_a(j) = 0 \);

Analogous problems for the Riemann-Liouville operator for \( \alpha > 1/p \) were solved in [9] (For boundedness two-weight criteria of general integral operators with positive kernels see [5], Chapter 3).

Let \( 0 < a \leq \infty \), \( k : \{(x, y) : 0 < y < x < a\} \to \mathbb{R}_+^1 \) be a kernel and let \( k_0(x) \equiv x k^2(x, x/2) \).
We denote by $L^p(\mathbb{L}_k^2(0, a))$ the set of all measurable functions $g : (0, a) \to \mathbb{R}^1$ for which
\[
\|g\|_{p(\mathbb{L}_k^2(0, \infty))} = \left( \sum_{n \in \mathbb{Z}} \left( \int_{2^n}^{2^{n+1}} |g(x)|^2k_0(x)dx \right)^{p/2} \right)^{1/p} < \infty
\]
if $a = \infty$ and
\[
\|g\|_{p(\mathbb{L}_k^2(0, a))} = \left( \sum_{n=0}^{+\infty} \left( \int_{2^{-n-a}}^{2^{-(n+1)-a}} |g(x)|^2k_0(x)dx \right)^{p/2} \right)^{1/p} < \infty
\]
if $a < \infty$, with the usual modification for $p = \infty$.

We shall need the following interpolation result (see, e.g., [19], p. 147 for the interpolation properties of the Schatten classes, and p. 127 for the corresponding properties of the sequence spaces. See also [1], Theorem 5.1.2):

**Proposition A.** Let $0 < a \leq \infty$, $1 \leq p_0, p_1 \leq \infty$, $0 \leq \theta \leq 1$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. If $T$ is a bounded operator from $l^{p_i}(\mathbb{L}_k^2(0, a))$ into $\sigma_{p_i}(\mathbb{L}^2(0, a))$, where $i = 0, 1$, then it is also bounded from $L^{p_i}(\mathbb{L}_k^2(0, a))$ into $\sigma_{p_i}(\mathbb{L}^2(0, a))$. Moreover,
\[
\|T\|_{l^{p_i}(\mathbb{L}_k^2(0, \infty)) \to \sigma_{p_i}(\mathbb{L}^2)} \leq \|T\|_{l^{p_0}(\mathbb{L}_k^2) \to \sigma_{p_0}(\mathbb{L}^2)}^{1-\theta}\|T\|_{l^{p_1}(\mathbb{L}_k^2) \to \sigma_{p_1}(\mathbb{L}^2)}^{\theta}.
\]

The next statement is obvious when $p = \infty$; and when $1 \leq p < \infty$ it follows from Lemma 2.11.12 of [15].

**Proposition B.** Let $1 \leq p \leq \infty$ and let $\{f_k\}$, $\{g_k\}$ be orthonormal systems in a Hilbert space $H$. If $T \in \sigma_p(H)$, then
\[
\|T\|_{\sigma_p(H)} \geq \left( \sum_{n} |\langle T f_n, g_n \rangle|^p \right)^{1/p}.
\]

Now we prove the main results.

In the sequel we shall assume that $v \in L^2_k(2^n, 2^{n+1})$ for all $n \in \mathbb{Z}$.

**Theorem 1.** Let $a = \infty$, $2 \leq p < \infty$ and let $k \in V \cap V_2$. Then $K_v$ belongs to $\sigma_p(\mathbb{L}^2(0, \infty))$ if and only if $v \in L^p(\mathbb{L}_k^2(0, \infty))$. Moreover, there exist positive constants $b_1$ and $b_2$ such that
\[
b_1\|v\|_{L^p(\mathbb{L}_k^2(0, \infty))} \leq \|K_v\|_{\sigma_p(L^2(0, \infty))} \leq b_2\|v\|_{L^p(\mathbb{L}_k^2(0, \infty))}.
\]
Proof. Sufficiency. Note that the fact $k \in V \cap V_2$ implies

$$I(x) \equiv \int_0^x k^2(x, y) dy \leq c k_0(x) \quad (2)$$

for some positive constant $c$ independent of $x$. Indeed, by the condition $k \in V \cap V_2$ we have

$$I(x) = \int_0^{x/2} k^2(x, y) dy + \int_{x/2}^x k^2(x, y) dy \leq c_1 k_0(x) + c_2 k_0(x) = c_3 k_0(x).$$

Consequently, using the Hilbert-Schmidt formula (1) and taking into account (2), we find that

$$\|Kv\|_{\sigma_2(L^2(0, \infty))} = \left( \int_0^\infty \int_0^x k^2(x, y) v^2(x) dx dy \right)^{1/2}$$

$$= \left( \int_0^x v^2(x) \left( \int_0^x k^2(x, y) dy \right) dx \right)^{1/2} \leq c_4 \left( \int_0^\infty v^2(x) k_0(x) dx \right)^{1/2}$$

$$= c_4 \left( \sum_{n \in \mathbb{Z}} \int_{2n}^{2n+1} v^2(x) k_0(x) dx \right)^{1/2} = c_4 \|v\|^2_{L_2^k(0, \infty)}.$$

On the other hand, in view of Theorem A we see that there exist positive constants $c_5$ and $c_6$ such that

$$c_5 \|v\|_{L_\infty(L_2^k(0, \infty))} \leq \|Kv\|_{\sigma_\infty(L^2(0, \infty))} \leq c_6 \|v\|_{L_\infty(L_2^k(0, \infty))}.$$

Further, Proposition A yields

$$\|Kv\|_{\sigma_p(L^2(0, \infty))} \leq c_7 \|v\|_{\sigma^*(L_2^k(0, \infty))},$$

where $2 \leq p < \infty$.

Necessity. Let $Kv \in \sigma_p(L^2(0, \infty))$ and let

$$f_n(x) = \chi_{[2n, 2n+1]}(x) 2^{-n/2},$$

$$g_n(x) = v(x)x^{1/2} \chi_{[3, 2n-1, 2n+1]}(x) k(x, x/2) \alpha_n^{-1/2},$$
where

\[ \alpha_n = \int_{3 \cdot 2^{n-1}}^{2^{n+1}} v^2(y) k_0(y) dy. \]

Then it is easy to verify that \( \{f_n\} \) and \( \{g_n\} \) are orthonormal systems. Further, by virtue of Proposition B (for \( p \geq 1 \)) we have

\[ \infty > \| K_v \|_{\sigma_p(L^2(0, \infty))} \geq \left( \sum_{n \in \mathbb{Z}} |\langle K_v f_n, g_n \rangle|^p \right)^{1/p} \]

\[ = \left( \sum_{n \in \mathbb{Z}} \left( \int_{3 \cdot 2^{n-1}}^{2^{n+1}} \left( \int_2^{x} 2^{-n/2} k(x, y) dy \right) v^2(x) x^{1/2} k(x, x/2) \alpha_n^{-1/2} dx \right)^p \right)^{1/p} \]

\[ \geq c_8 \left( \sum_{n \in \mathbb{Z}} \left( \alpha_n^{-1/2} \int_{3 \cdot 2^{n-1}}^{2^{n+1}} 2^{-n/2} k(x, x/2) v^2(x) (x - 2^n) x^{1/2} dx \right)^p \right)^{1/p} \]

\[ \geq c_9 \left( \sum_{n \in \mathbb{Z}} \left( \alpha_n^{-1/2} \int_{3 \cdot 2^{n-1}}^{2^{n+1}} k_0(x) v^2(x) dx \right)^p \right)^{1/p} = c_9 \left( \sum_{n \in \mathbb{Z}} \alpha_n^{p/2} \right)^{1/p}. \]

Now let

\[ f'_n(x) = \chi_{[3 \cdot 2^{n-2}, 3 \cdot 2^{n-1}]}(x)(3 \cdot 2^{n-2})^{-1/2} \]

and

\[ g'_n(x) = v(x) x^{1/2} \chi_{[2^n, 3 \cdot 2^{n-1}]}(x) k(x, x/2) \beta_n^{-1/2}, \]

where

\[ \beta_n = \int_{2^n}^{3 \cdot 2^{n-1}} v^2(y) k_0(y) dy. \]

Then it is easy to verify that \( \{f'_n\} \) and \( \{g'_n\} \) are orthonormal systems. Further,

\[ \infty > \| K_v \|_{\sigma_p(L^2(0, \infty))} \geq \left( \sum_{n \in \mathbb{Z}} |\langle K_v f'_n, g'_n \rangle|^p \right)^{1/p} \]

\[ = \left( \sum_{n \in \mathbb{Z}} \left( \int_{3 \cdot 2^{n-2}}^{2 \cdot 2^{n-1}} \left( \int_{3 \cdot 2^{n-2}}^{x} (3 \cdot 2^{n-2})^{-1/2} k(x, y) dy \right) v^2(x) x^{1/2} k(x, x/2) \beta_n^{-1/2} dx \right)^p \right)^{1/p} \]
\[ \geq c_{10} \left( \sum_{n \in \mathbb{Z}} \left( \beta_n^{-1/2} \int_{2^n}^{2^{n+1}} 2^{-(n-2)/2} k^2(x, x/2) v^2(x) \right)^p \right)^{1/p} \]
\[ \times (x - 3 \cdot 2^{n-2})^{1/2} dx \right) \right)^{1/p} \]
\[ \geq c_{11} \left( \sum_{n \in \mathbb{Z}} \left( \beta_n^{-1/2} \int_{2^n}^{2^{n+1}} k_0(x) v^2(x) dx \right)^p \right)^{1/p} \]
\[ = c_{11} \left( \sum_{n \in \mathbb{Z}} \beta_n^{p/2} \right)^{1/p}, \]
where \( p \geq 1 \). Consequently
\[ \left( \sum_{n \in \mathbb{Z}} \left( \int_{2^n}^{2^{n+1}} v^2(x) k_0(x) dx \right)^{p/2} \right)^{1/p} \leq \left( \sum_{n \in \mathbb{Z}} (\beta_n + \alpha_n)^{p/2} \right)^{1/p} \]
\[ \leq c_{12} \|K_v\|_{\sigma_p(L^2(0, \infty))} + c_{12} \|K_v\|_{\sigma_p(L^2(0, \infty))} \]
\[ \leq c_{13} \|K_v\|_{\sigma_p(L^2(0, \infty))} < \infty. \]

Let us now consider the case \( a < \infty \). We have the following statement:

**Theorem 2.** Let \( 0 < a < \infty, 2 \leq p < \infty \) and let \( k \in V \cap V_2 \). Then \( K_v \) belongs to \( \sigma_p(L^2(0, a)) \) if and only if \( v \in L^p(L^2_{k_0}(0, a)) \). Moreover, there exists positive constants \( b_1 \) and \( b_2 \) such that
\[ b_1 \|v\|_{L^p_k(0, a)} \leq \|K_v\|_{\sigma_p(L^2(0, a))} \leq b_2 \|v\|_{L^p_k(0, a)}. \]

**Proof.** **Sufficiency.** The Hilbert–Schmidt formula and the condition \( k \in V \cap V_2 \) yield
\[ \|K_v\|_{\sigma_p(L^2(0, a))} = \left( \int_0^a v^2(x) \left( \int_0^x k^2(x, y) dy \right) dx \right)^{1/2} \]
\[ \leq c_1 \left( \int_0^a v^2(x) k_0(x) dx \right)^{1/2} \]
\[ = c_1 \left( \sum_{n=0}^{\infty} \int_{2^{-(n+1)}}^{2^{-n}} v^2(x) k_0(x) dx \right)^{1/2} \]
\[ = c_1 \|v\|_{L^2_k(0, a)}. \]
In view of Theorem B (part (a)) we arrive at
\[ \|K_v\|_{\sigma_\infty(L^2(0,a))} \approx \|v\|_{l_\infty(L^2_{k_0}(0,a))}. \]
Using Proposition A we derive
\[ \|K_v\|_{\sigma_p(L^2(0,a))} \leq c_2 \|v\|_{l_p(L^2_{k_0}(0,a))} \]
when \( p \geq 2 \).
To prove necessity we take the orthonormal systems of functions defined on \((0,a)\):
\[ f_n(x) = \chi_{[2^{-(n+1)a},2^{-n}a]}(x)(2^{-(n+1)a})^{-1/2} \]
and
\[ g_n(x) = v(x)x^{1/2} \chi_{[3 \cdot 2^{-(n+2)},2^{-n}a]}(x)k(x,x/2)\alpha_n^{-1/2}, \]
where
\[ \alpha_n = \int_{3 \cdot 2^{-(n+2)a}}^{2^{-n}a} v^2(y)k_0(y)dy \]
and \( n = 0, 1, 2, \ldots \). Consequently Proposition B yields
\[ \infty > \|K_v\|_{\sigma_p(L^2(0,a))} \geq \left( \sum_{n=0}^{+\infty} |\langle K_v f_n, g_n \rangle|^p \right)^{1/p} \]
\[ = \left( \sum_{n=0}^{+\infty} \left( \int_{3 \cdot 2^{-(n+2)a}}^{2^{-n}a} x^{1/2} v^2(x)k(x,x/2) \right) \right)^{1/p} \]
\[ \times \left( \int_{2^{-(n+1)a}}^{x} (2^{-(n+1)a})^{-1/2} k(x,y)dy \right)^{p/2} \]
\[ \geq c_3 \left( \sum_{n=0}^{+\infty} \alpha_n^{p/2} \right)^{1/p}. \]
If we take the following orthonormal systems:
\[ f'_n(x) = \chi_{[3 \cdot 2^{-(n+3)a},3 \cdot 2^{-(n+2)a}]}(x)(3 \cdot 2^{-(n+3)a})^{-1/2}, \]
\[ g'_n(x) = v(x)x^{1/2} \chi_{[2^{-(n+1)a},3 \cdot 2^{-(n+2)a}]}(x)k(x,x/2)\beta_n^{-1/2}, \]
where
\[ \beta_n = \int_{2^{-(n+1)}}^{3-2^{-(n+2)}} v^2(y)k_0(y)dy, \]
then we arrive at the estimate
\[ \|K_v\|_{\sigma_p(L^2(0,a))} \geq c_4 \left( \sum_{n=0}^{\infty} \beta_n^{p/2} \right)^{1/p}. \]
Finally we have the lower estimate for \( \|K_v\|_{\sigma_p(L^2(0,a))} \).

**Remark 1.** It follows from the proof of Theorems 1 and 2 that the lower estimate of \( \|K_v\|_{\sigma_p(L^2(0,a))} \) holds for \( 1 \leq p \leq \infty \).

Now we formulate and prove the next statement.

**Proposition 1.** Let \( 1 \leq p < \infty \). Then
\[ \|v\|_{L^p(L^2_k(0,\infty))} \approx J(v,p), \]
where
\[ J(v,p) = \left( \int_0^\infty \left( \int_{x/2}^{2x} v^2(y)k^2(y,y/2)dy \right)^{p/2} x^{p/2-1}dx \right)^{1/p}. \]

**Proof.** We have
\[ \|v\|_{L^p(L^2_k(0,\infty))} = \left( \sum_{n \in \mathbb{Z}} \left( \int_{2^n}^{2^{n+1}} v^2(x)k_0(x)dx \right)^{p/2} \right)^{1/p} \]
\[ \leq \left( \sum_{n \in \mathbb{Z}} \left( \int_{2^n}^{2^{n+1}} v^2(x)k^2(x,x/2)dx \right)^{p/2} \right)^{1/p}, \]
\[ = c_1 \left( \sum_{n \in \mathbb{Z}} \left( \int_{2^n}^{2^{n+1}} v^2(x)k^2(x,x/2)dx \right)^{p/2} \right)^{1/p} \]
\[ \leq c_2 \left( \sum_{n \in \mathbb{Z}} \int_{2^n}^{2^{n+1}} y^{p/2-1} \left( \int_{2^n}^{2^{n+1}} v^2(x)k^2(x,x/2)dx \right)^{p/2} dy \right)^{1/p} \]
\[ \leq c_2 \left( \sum_{n \in \mathbb{Z}} \int_{2^n}^{2^{n+1}} y^{p/2-1} \left( \int_{y/2}^{2y} v^2(x)k^2(x,x/2)dx \right)^{p/2} dy \right)^{1/p} = c_2 J(v,p). \]
To prove the reverse inequality we observe that

\[
J(v, p) = \left( \sum_{n \in \mathbb{Z}} \int_{2^n}^{2^{n+1}} y^{p/2-1} \left( \int_{y/2}^{2y} v^2(x) k^2(x, x/2) dx \right)^{p/2} dy \right)^{1/p} \\
\leq \left( \sum_{n \in \mathbb{Z}} \left( \int_{2^n}^{2^{n+1}} y^{p/2-1} dy \right) \left( \int_{2^{n-1}}^{2^n} v^2(x) k^2(x, x/2) dx \right)^{p/2} \right)^{1/p} \\
\leq c_3 \left( \sum_{n \in \mathbb{Z}} 2^{np/2} \left( \int_{2^{n-1}}^{2^n} v^2(x) k^2(x, x/2) dx \right)^{p/2} \right)^{1/p} \\
+ c_3 \left( \sum_{n \in \mathbb{Z}} 2^{np/2} \left( \int_{2^n}^{2^{n+1}} v^2(x) k^2(x, x/2) dx \right)^{p/2} \right)^{1/p} \\
+ c_3 \left( \sum_{n \in \mathbb{Z}} 2^{np/2} \left( \int_{2^n}^{2^{n+1}} v^2(x) k^2(x, x/2) dx \right)^{p/2} \right)^{1/p} \\
\leq c_4 \|v\|_{L^p(L^2(0, \infty))}.
\]

From Theorem 1 and Proposition 1 we easily derive the following statement:

**Theorem 3.** Let \(2 \leq p < \infty\) and let \(k \in V \cap V_\lambda\). Then

\[
\|Kv\|_{\sigma_p(L^2(0, \infty))} \approx J(v, p).
\]

A result analogous to Theorem 1 was obtained in [13] for the Riemann-Liouville operator \(R_{\alpha,v}\), assuming that \(\alpha > 1/2\) and \(p > 1/\alpha\) (see [14] for \(\alpha = 1\) and \(p > 1\)).

Let us now consider the multidimensional case. In particular, we shall deal with the operator

\[
B_{+,v}^\alpha f(x) = v(x) \int_{|y| < |x|} \left( \frac{|x|^2 - |y|^2}{|x - y|^4} \right)^\alpha f(y) dy, \quad \alpha > 0,
\]

where \(v\) is a Lebesgue-measurable function on \(\mathbb{R}^n\) with \(v \in L^2(\{2^n < |y| < 2^{n+1}\})\) for all \(n \in \mathbb{Z}\) (for the definition and some properties of \(B_{+,v}\), where \(v \equiv 1\), see, e.g., [16], Chapter 7, and [17], Section 29).
Let $w$ be a measurable a.e. positive function on $\mathbb{R}^n$. We denote by $l^p(L^2_w(\mathbb{R}^n))$ a set of all measurable functions $\varphi: \mathbb{R}^n \to \mathbb{R}$ for which

$$
\|\varphi\|_{l^p(L^2_w(\mathbb{R}^n))} = \left( \sum_{k \in \mathbb{Z}} \left( \int_{2^k < |x| < 2^{k+1}} \varphi^2(x)w(x)dx \right)^{p/2} \right)^{1/p} < \infty.
$$

The next result is from [19] (pp. 127, 147).

**Proposition C.** Let $1 \leq p_0, p_1 \leq \infty$, $0 \leq \theta \leq 1$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. If $T$ is a bounded operator from $l^p(L^2_w(\mathbb{R}^n))$ into $\sigma_{p_i}(L^2_w(\mathbb{R}^n))$, where $i = 0, 1$, then it is also bounded from $l^p(L^2_w(\mathbb{R}^n))$ into $\sigma_{p_i}(L^2(\mathbb{R}^n))$.

First we formulate some statements concerning the mapping properties of $B_{\alpha,v}^{\alpha}$.

**Theorem C ([12]).** Let $1 < p \leq q < \infty$, $\alpha > \frac{n}{p}$. Then $B_{\alpha,v}^{\alpha}$ acts boundedly from $L^p(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$ if and only if

$$
F \equiv \sup_{j \in \mathbb{Z}} F(j) \equiv \sup_{j \in \mathbb{Z}} \left( \int_{2^j < |x| < 2^{j+1}} |v(x)|^q |x|^{2\alpha-n/p}dx \right)^{1/q} < \infty.
$$

Moreover, $\|B_{\alpha,v}^{\alpha}\| \approx F$.

The following result can be obtained in the same as Theorem 5 from [12], therefore we omit the proof (see also [11]).

**Theorem D.** Let $1 < p \leq q < \infty$ and let $\alpha > \frac{n}{p}$. Then $B_{\alpha,v}^{\alpha}$ acts compactly from $L^p(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$ if and only if $F < \infty$ and $\lim_{j \to +\infty} F(j) = 0$.

Now we state and prove the following Theorem:

**Theorem 4.** Let $2 \leq p < \infty$ and let $\alpha > n/2$. Then $B_{\alpha,v}^{\alpha} \in \sigma_{p}(L^2(\mathbb{R}^n))$ if and only if $v \in \sigma_p(L^2_{4\alpha-n}(\mathbb{R}^n))$. Moreover, there exist positive constants $b_1$ and $b_2$ such that

$$
b_1 \|v\|_{\sigma_p(L^2_{4\alpha-n}(\mathbb{R}^n))} \leq \|B_{\alpha,v}^{\alpha}\|_{\sigma_p(L^2(\mathbb{R}^n))} \leq b_2 \|v\|_{\sigma_p(L^2_{4\alpha-n}(\mathbb{R}^n))}.
$$
Proof. For sufficiency, we use the Hilbert-Schmidt formula (1) and the condition $\alpha > \frac{n}{2}$. Thus,

$$\|B_{+v}^{\alpha}\|_{\sigma_2(L^2(\mathbb{R}^n))} = \left( \int_{\mathbb{R}^n} v^2(x) \left( \int_{|y|<|x|} \left( \frac{|x|^2 - |y|^2}{|x-y|^{2n}} \right)^{2\alpha} dy \right) dx \right)^{\frac{1}{2}}$$

$$\leq c_1 \left( \int_{\mathbb{R}^n} |x|^{2\alpha} v^2(x) \left( \int_{|y|<|x|} |x-y|^{-(\alpha-n)2} dy \right) dx \right)^{\frac{1}{2}}$$

$$\leq c_2 \left( \int_{\mathbb{R}^n} |x|^{4\alpha-n} v^2(x) dx \right)^{\frac{1}{2}} = c_2 \left( \sum_{k=-\infty}^{+\infty} a_k^2 \right)^{\frac{1}{2}},$$

where

$$a_k = \left( \int_{2^k<|y|<2^{k+1}} |x|^{4\alpha-n} v^2(x) dx \right)^{1/2}.$$  

Moreover, using Theorem C we arrive at the following two-sided inequality:

$$c_3 \|v\|_{l^\infty(L^2_{4\alpha-n}(\mathbb{R}^n))} \leq \|B_{+v}^{\alpha}\|_{\sigma_\infty(L^2(\mathbb{R}^n))} \leq c_4 \|v\|_{l^\infty(L^2_{4\alpha-n}(\mathbb{R}^n))}.$$  

By Proposition C we conclude that

$$\|B_{+v}^{\alpha}\|_{\sigma_p(L^2(\mathbb{R}^n))} \leq c_5 \|v\|_{l^p(L^2_{4\alpha-n}(\mathbb{R}^n))}, \quad 2 \leq p < \infty.$$  

Now we prove necessity. For this we take the orthonormal systems $\{f_k\}$ and $\{g_k\}$, where

$$f_k(x) = \chi\{2^{k-2}<|y|<2^{k-1}\}(x)2^{-(k-2)n/2} \cdot \lambda_n^{-\frac{1}{2}},$$

$$g_k(x) = \chi\{2^k\leq|y|<2^{k+1}\}(x) |x|^{2\alpha-n} \cdot \frac{\lambda_n^{-\frac{1}{2}}}{v(x)}.$$  

$$\lambda_n = (2^n - 1)\pi^{n/2}/\Gamma(n/2 + 1)$$  

and

$$\alpha_k = \int_{2^k<|x|<2^{k+1}} v^2(x) |x|^{4\alpha-n} dx.$$  

Then in view of Proposition B we have
\[
\begin{align*}
\infty > \| B_+^\alpha v \|_{\sigma_p(L^2(\mathbb{R}^n))} & \geq c_6 \left( \sum_{k \in \mathbb{Z}} \left( \int_{2^k < |x| < 2^{k+1}} v^2(x) |x|^{2\alpha - \frac{n}{2}} \right) \right)^{1/2} \\
& \times \left( \sum_{2^{k-2} < |y| < 2^{k-1}} \left( \frac{|x|^2 - |y|^2 \alpha}{|x - y|^n} 2^{-(k-2)n/2} \right) \right)^{1/2} \\
& \geq c_7 \left( \sum_{k \in \mathbb{Z}} \alpha^{p/2} \right)^{1/p} = c_7 \| v \|_{\sigma_p(L^2_{4\alpha-n}(\mathbb{R}^n))}.
\end{align*}
\]
which completes the proof.

The following result is also true:

**Theorem 5.** Let $2 \leq p < \infty$ and let $\alpha > n/2$. Then $B_+^\alpha v \in \sigma_p(L^2(\mathbb{R}^n))$ if and only if
\[
\left( \int_{\mathbb{R}^n} \left( \int_{|x|^2 < |y|^n} v^2(y) |y|^{4\alpha - 2n} \right)^{p/2} |x|^{np/2 - n} \right)^{1/p} < \infty.
\]
Moreover,
\[
c_1 I(v, p, \alpha) \leq \| B_+^\alpha v \|_{\sigma_p(L^2(\mathbb{R}^n))} \leq c_2 I(v, p, \alpha)
\]
for some positive constants $c_1$ and $c_2$.

**Proof.** Taking into account Theorem 4, the statement will be proved if we show that
\[
\| v \|_{\sigma_p(L^2_{4\alpha-n}(\mathbb{R}^n))} \approx I(v, p, \alpha).
\]
Indeed, we have
\[
\begin{align*}
\| v \|_{\sigma_p(L^2_{4\alpha-n}(\mathbb{R}^n))} & \leq \left( \sum_{k \in \mathbb{Z}} \left( \int_{2^k < |x| < 2^{k+1}} v^2(x) |x|^{4\alpha - 2n} \right)^{p/2} 2^{(k+1)np/2} \right)^{1/p} \\
& = b_1 \left( \sum_{k \in \mathbb{Z}} \int_{2^k < |y| < 2^{k+1}} |y|^{np/2 - n} \left( \int_{2^k < |x| < 2^{k+1}} v^2(x) |x|^{4\alpha - 2n} \right)^{p/2} \right)^{1/p}.
\end{align*}
\]
\[ = b_1 I(v, p, \alpha). \]

The reverse inequality follows similarly.

**Remark 2.** Some results of this paper were announced in [11].

**Acknowledgement.** The work was supported by Grant No. 1.7 of the Georgian Academy of Sciences.

I express my gratitude to Prof. V. Kokilashvili for drawing my attention to the above-considered problems and to the referee for helpful remarks and comments.

**References**


A. Razmadze Mathematical Institute
Georgian Academy of Sciences
1, M. Aleksidze St., Tbilisi 380093
Georgia
E-mail: meskhi@rmi.acnet.ge

Recibido: 5 de Enero de 2001
Revisado: 6 de Marzo de 2001