

A VOLUME COMPARISON THEOREM AND NUMBER OF ENDS FOR MANIFOLDS WITH ASYMPTOTICALLY NONNEGATIVE RICCI CURVATURE

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Abstract

In this paper we establish a volume comparison theorem for concentric metric balls at arbitrary point for manifolds with asymptotically nonnegative Ricci curvature, which will allow us to prove the finiteness of the number of ends

1 Introduction

In (CG1) and (CG2) J.Cheeger and D.Gromoll studied noncompact manifolds with nonnegative (sectional and Ricci) curvature and showed they have finite topology.

In 1985 U.Abresch in [Ab1] and [Ab2] introduced a new concept: manifolds with asymptotically nonnegative sectional curvature. A noncompact manifold of dimension m is said to be with asymptotically nonnegative sectional curvature (Ricci curvature) if there exists a point p , called base point and a positive, nonincreasing function λ so that:

$$\int_0^{+\infty} t\lambda(t)dt = b_0 < +\infty$$

and for all $x \in M$ and u in $T_x M$

$$K(u) \geq -\lambda(d(p, x))\|u\|^2 \quad (\text{Ricci}(u) \geq -(m-1)\lambda(d(p, x))\|u\|^2).$$

One of the remaining questions is if asymptotically nonnegative curvature implies finiteness of number of ends. U.Abresch and D.Gromoll in

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[AG] showed that if M has asymptotically nonnegative curvature and has diameter growth of order $o(r^{\frac{1}{n}})$ then it is homotopically equivalent to an interior of compact manifold with boundary, provided the sectional curvature is bounded from below. M.I.Cai in [Ca] established a bound on number of ends of manifolds with nonnegative Ricci curvature outside a compact subset. S.H.Zhu in [Zh] proved a volume comparison theorem for balls with center the base point in manifolds with asymptotically nonnegative Ricci curvature.

The purpose of this paper is to establish a comparison theorem at every point, not necessary the base point and deduce a finite bound on the number of ends. In the second theorem we show that manifolds with asymptotically nonnegative Ricci curvature have infinite volume.

Let take some definitions .

Two curves γ_1 and γ_2 on a riemannian manifold M with origin x are said to be cofinal if and only if for all ball $B(x,r)$, there exists $t > 0$ so that $\gamma_1(t_1)$ and $\gamma_2(t_2)$ lie in the same connected component of $M/\overline{B(x,r)}$ for all $t_1, t_2 \geq t$ where $\overline{B(x,r)}$ denotes the closed ball of center x and radius r .

In this way, an equivalence relation is defined and every class is called an end of M .

Let $r > 0$ and $B(p,r)$ the ball of center p and radius r ; we note $\mathcal{C}(p,r)$ the union of all connected unbounded components of $M/\overline{B(p,r)}$. Let ζ be a real, $\frac{1}{2} < \zeta < 1$, Ω an open set in M and Σ a connected subset of Ω ; for $x, y \in \Sigma \cap S(p,r)$ let:

$$d_r(x, y) = \inf_{\gamma} L(\gamma),$$

where infimum is taken over all smooth curves γ in Σ from x to y . We set

$$\text{diam}(\Sigma \cap S(p,r), \Omega) = \sup_{(x,y) \in \Sigma} d_r(x, y).$$

and

$$\text{Diam}(p,r) = \text{Supdiam}(\Sigma \cap S(p,r), \mathcal{C}(p, \zeta r)),$$

where supremum is taken over all unbounded connected components Σ of $\partial\mathcal{C}(p,r)$. The function $\text{Diam}(p,r)$ measures the diameter of ends.

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2 Main Results

Theorem 1. *Let M be a complete open Riemannian manifold with dimension m and asymptotically nonnegative Ricci curvature. Then, for all point x in M and for all $0 < r \leq R$*

$$\frac{\text{Vol}(B(x, R))}{\text{Vol}(B(x, r))} \leq e^{(m-1)b_0} \left(\frac{R}{r}\right)^m \text{ if } 0 \leq R \leq l$$

$$\frac{\text{Vol}(B(x, R))}{\text{Vol}(B(x, r))} \leq e^{(m-1)b_0} \left(\frac{R+l}{r}\right)^m \text{ if } R \geq l$$

where $B(x, r)$ denotes the ball of radius r with center x in M .

Corollary. *Let M be a manifold with asymptotically nonnegative Ricci curvature with base point p . Then,*

$$\text{Diam}(p, r) \leq 4\xi e^{(m-1)b_0} \left(1 + \frac{3}{\xi}\right) r.$$

where $\xi = \frac{1}{2}(1 - \zeta)$ and consequently M has finite number of ends. This corollary shows that the diameter of ends of an asymptotically nonnegative curved manifold grown at most lineary as on nonnegative curved manifolds.

Theorem 2. *Let M be a manifold with asymptotically nonnegative Ricci curvature with base point p . Then, there exist two positive constants C and ρ such that, for all $R > 0$*

$$\text{Vol}(B(p, R)) \geq C(\ln R)^\rho.$$

For all $t \geq 0$, let $\alpha(t) = \lambda(|t - l|)$ where $l = d(p, x)$. If $z \in M$ and $u \in T_z M$ then, by triangle inequality, we have:

$$|d(x, z) - d(x, p)| \leq d(p, z)$$

and since λ is nonincreasing ,

$$\text{Ric}_z(u) \geq -(m-1)\lambda(d(p, z)) \geq -(m-1)\alpha(d(x, z))$$

3 Proofs

To prove our results we need three lemmas:

Lemma1. Let $y(t)$ be the unique solution of equation:

$$(*) \begin{cases} y''(t) - \alpha(t)y(t) = 0 \\ y(0) = 0, y'(0) = 1 \end{cases}$$

then for all $t \geq 0$, $t \leq y(t) \leq ce^{b_0 \cdot t}$ where

$$c = \begin{cases} 1 & \text{if } t \leq l \\ (1 + 2l\lambda_1(0))e^{b_0} & \text{if } t \geq l \end{cases}$$

where $\lambda_1(0) = \int_0^\infty \lambda(x)dx$

Proof. Since $y''(t) = \alpha(t)y(t) \geq 0$, y' is increasing in a neighbourhood of zero and one easily shows like in [Zh] that $y(t) \geq t$ for all $t \geq 0$ and that y is increasing.

By initial conditions, we have:

$$y'(t) = 1 + \int_0^t \alpha(s)y(s)ds \implies$$

$$y(t) = t + \int_0^t \left(\int_0^s \alpha(x)y(x)dx \right) ds = t + \int_0^t \left(\int_x^t \alpha(x)y(x)ds \right) dx$$

by Fubini.

$$y(t) = t + \int_0^t (t-x)\alpha(x)y(x)dx = t \left(1 + \int_0^t \alpha(x)y(x)dx \right) - \int_0^t x\alpha(x)y(x)dx$$

$$\implies y(t) \geq t \left(1 + \int_0^t \alpha(x)y(x)dx \right) - ty(t) \int_0^t \alpha(x)dx$$

$$\implies 1 + t \int_0^t \alpha(x)dx \geq \frac{ty'(t)}{y(t)} (**)$$

If $l = 0$ inequality holds by theorem2.1 in [Zh]

Suppose $l > 0$.

If $t \leq l$ then $(**)$ implies

$$\frac{1}{t} + \int_0^t \alpha(x)dx \geq \frac{y'(t)}{y(t)}.$$

After integrating this expression we have

$$\begin{aligned} &\implies \ln \frac{t}{\epsilon} + \int_{\epsilon}^t \int_0^x \lambda(l-u) du dx \geq \ln \frac{y(t)}{y(\epsilon)} \\ &\implies \ln y(t) \leq \ln \frac{ty(\epsilon)}{\epsilon} + \int_{\epsilon}^t \int_0^x \lambda(l-u) du dx \end{aligned}$$

By the initial conditions, we have

$$\begin{aligned} \ln y(t) &\leq \ln t + \int_{\epsilon}^t \int_0^x \lambda(l-u) du dx \leq \ln y(t) \leq \ln t + \int_0^t \int_0^x \lambda(l-u) du dx \\ &\implies \ln y(t) \leq \ln t + \int_0^t \int_u^t dx \lambda(l-u) du \\ &\implies \ln y(t) \leq \ln t + \int_0^t (t-u) \lambda(l-u) du \leq \ln t + \int_0^l u \lambda(u) du \\ &\leq \ln t + b_0 \end{aligned}$$

that is

$$y(t) \leq e^{b_0} \cdot t$$

If $t \geq l$ we have:

$$\begin{aligned} y'(t) &= y'(l) + \int_l^t \alpha(x) y(x) dx \\ \implies y(t) - y(l) &= (t-l)y'(l) + \int_l^t \int_l^x y(u) \lambda(u-l) du dx \\ &= (t-l)y'(l) + \int_l^t (t-u) \lambda(u-l) y(u) du \\ &\leq (t-l) \left(y'(l) + \int_l^t \lambda(u-l) y(u) du \right) = (t-l)y'(t). \end{aligned}$$

then

$$\frac{\alpha(t)y(t)}{y'(t)} \leq (t-l)\alpha(t) + \frac{\alpha(t)y(l)}{y'(t)}$$

hence

$$\ln \frac{y'(t)}{y'(l)} \leq \int_l^t (x-l)\alpha(x) + \int_l^t \frac{\alpha(x)y(l) dx}{y'(l) + \int_l^x \alpha(u)y(u) du}$$

$$\ln \frac{y'(t)}{y'(l)} \leq b_0 + \int_l^t \frac{\alpha(x)y(l)dx}{y'(l) + y(l) \int_l^x \alpha(u)du}$$

$$\ln \left(\frac{y'(t)}{y'(l)} \right) \leq b_0 + \ln \left(\frac{y'(l) + y(l)\lambda_1(0)}{y'(l)} \right)$$

From (**) we have

$$y'(l) \leq y(l) \left(\frac{1}{l} + \lambda_1(0) \right) \leq e^{b_0} (1 + l\lambda_1(0))$$

$$\implies y'(t) \leq e^{2b_0} \cdot (1 + 2l\lambda_1(0))$$

hence

$$y(t) \leq e^{2b_0} (1 + 2l\lambda_1(0))t$$

and the lemma follows. The lemma is far from giving a sharp comparison for volume of balls with center $x \neq p$ and a sufficient large radius. The basic fact on this is related to the choice of the function α . So, we state the following lemma to prove our comparison theorem:

Lemma 2. *Let \overline{M} be a noncompact simply connected manifold with dimension m . Suppose there exists a point \overline{p} so that*

$$K(\overline{x}) = -\lambda(d(\overline{p}, \overline{x}))$$

for all $\overline{x} \in \overline{M}$. Then, for all $R > 0$ and \overline{x} , we have 0

$$(1) \quad Vol(B(\overline{p}, R)) \leq \omega_m R^m e^{(m-1)b_0}$$

$$(2) \quad Vol(B(\overline{x}, r)) \geq \omega_m r^m$$

and consequently

$$(3) \quad \frac{Vol(B(\overline{x}, R))}{Vol(B(\overline{x}, r))} \leq \begin{cases} e^{(n-1)b_0} \left(\frac{R}{r} \right)^m & \text{if } r \leq R \leq l \\ e^{(n-1)b_0} \left(\frac{R+l}{r} \right)^m & \text{if } R \geq l \end{cases}$$

where ω_m is the volume of the unit ball in euclidian space.

Proof

For all $R > 0$ we have

$$Vol(B(\bar{p}, R)) = \int_{S^{n-1}} \int_0^R y^{m-1}(t) dt d\theta$$

By lemma 1 we have $y(t) \leq e^{b_0 t} (l = 0)$ and the conclusion follows. Since $K(\bar{x}) \leq 0$ for all \bar{x} , the inequality (2) follows from Rauch inequality and the fact that \bar{M} is simply connected. .

If $R \geq l$ we have

$$B(\bar{x}, R) \subset B(\bar{p}, R + l)$$

and inequality (3) follows from (1) and (2).

The following lemma was proved in [Zh] (lemma2.2)

Lemma 3. *Let f and g be two positive functions defined over $[0, +\infty[$. If f/g is nonincreasing, then for $R \geq r > 0$ we have:*

$$\frac{\int_0^R f(t) dt}{\int_0^r f(t) dt} \leq \frac{\int_0^R g(t) dt}{\int_0^r g(t) dt}.$$

Proof of theorem 1

Let J denote the Jacobian of exponential application in polar coordinates and the nonnegative function z so that $z^{m-1} = J$; J. Cheeger showed in [Ch] that z satisfies the inequation:

$$\begin{cases} z''(t) - \alpha(t)z(t) \leq 0 \\ z(0) = 0, z'(0) = 1 \end{cases}$$

from which it follows that $\frac{z}{y}$ is nonincreasing where y is the solution of (*) and by the lemma2 we conclude that

$$\begin{aligned} \frac{Vol(B(x, R))}{Vol(B(x, r))} &= \frac{\int_0^{\min\{cut(\theta), R\}} J(t) dt}{\int_0^{\min\{cut(\theta), r\}} J(t) dt} \leq \frac{\int_0^{\min\{cut(\theta), R\}} y^{m-1}(t) dt}{\int_0^{\min\{cut(\theta), r\}} y^{m-1}(t) dt} \\ &\leq \frac{\int_0^R y^{m-1}(t) dt}{\int_0^r y^{m-1}(t) dt} = \frac{Vol(B(\bar{x}, R))}{Vol(B(\bar{x}, r))} \end{aligned}$$

$$\leq e^{(m-1)b_0} \left(\frac{R+l}{r} \right)^m$$

where $cut(\theta)$ means the cut point of θ

Proof of corollary

If $r > 0$ and $\{q_j\}$ is the maximal set of points on $S(p, r)$ so that the balls $B(q_j, \xi r)$ are disjoint and are contain in $M/B(p, \zeta r)$. We have :

$$B(q_j, \xi.r) \subset B(p, (1 + \xi).r) \subset B(q_j, (2 + \xi).r)$$

then

$$\frac{Vol(B(p, (1 + \xi).r))}{Vol(B(q_j, \xi.r))} \leq \frac{Vol(B(q_j, (2 + \xi).r))}{Vol(B(q_j, \xi.r))} \leq e^{(m-1)b_0} \left(\frac{3 + \xi}{\xi} \right)^m,$$

therefore, the number of balls $B(q_j, \xi r)$ is no more than $e^{(m-1)b_0} \left(1 + \frac{3}{\xi} \right)^m$.

The balls $B(q_j, 2\xi.r)$ cover $S(p, r)$ and if γ is a geodesic in $S(p, r)$ joining two points q_i and q_j so that $B(q_j, 2\xi.r) \cap B(q_i, 2\xi.r)$ is not empty then $L(\gamma) \leq 4\xi r$, hence

$$Diam(p, r) \leq 4\xi e^{(m-1)b_0} \left(1 + \frac{3}{\xi} \right)^m r.$$

Let $\{\gamma_i\}$ be the set of all geodesics from the base point p ; if $x_1 = \gamma_1((1 + \xi)r), x_2 = (\gamma_2((1 + \xi)r)$ are in two different connected components of $\overline{M/B(p, r)}$ and if θ is a short geodesic joining x_1 to x_2 , then θ meets $\overline{B(p, r)}$ and

$$\begin{aligned} d(\gamma_1(1 + \xi)r, \gamma_2(1 + \xi)r) &\geq d(\gamma_1(1 + \xi)r, B(p, r)) + d(B(p, r), \gamma_2(1 + \xi)r) \\ &\geq 2\xi.r, \end{aligned}$$

hence the ball $\overline{B(q_j, \xi.r)}$ contains at most one $\gamma_k(1 + \xi r)$ which means that the number of ends is less or equal to $e^{(m-1)b_0} \left(\frac{3+\xi}{\xi} \right)^m$.

Theorem2 states that, as for open complete manifolds with nonnegative Ricci curvature, the open complete manifolds with asymptotically nonnegative Ricci curvature have no finite volume.

Proof of theorem 2

Since M is noncompact and complete, for all $t > 0$ there exists a point x in M so that $d(p, x) \geq t$. Let γ be geodesic arc joining p to x , the function $s \mapsto d(p, \gamma(s))$ is continuous and takes the value t . Let $b > 1$, $R_i = \sum_{j=0}^i 2r_j$ and $r_i = 2^{b^i}$. Take a point x_i on γ so that $d(p, x_i) = r_i + R_{i-1}$ and $x_0 = p$; by construction the balls $B(x_i, r_i)$ are disjoint and

$$\bigcup_{j=0}^i B(x_j, R_j) \subset B(x_i, r_i + R_{i-1}).$$

Let

$$\theta_i = \sum_{j=0}^i \text{vol}B(x_j, r_j);$$

then

$$\begin{aligned} \theta_i &\leq \text{vol}B(x_i, r_i + R_{i-1}) \\ \frac{\theta_i}{\theta_i - \theta_{i-1}} &\leq \frac{\text{vol}B(x_i, r_i + R_{i-1})}{\text{vol}B(x_i, r_i)} \leq e^{(m-1)b_0} \left(1 + \frac{R_{i-1}}{r_i}\right)^m \end{aligned}$$

since $d(p, x_i) = r_i + R_{i-1} = l$ hence

$$\theta_i \geq \frac{e^{(m-1)b_0} \left(1 + \frac{R_{i-1}}{r_i}\right)^m}{e^{(m-1)b_0} \left(1 + \frac{R_{i-1}}{r_i}\right)^m - 1} \theta_{i-1}$$

$R_{i-1} = \sum_{j=0}^{i-1} 2 \cdot 2^{bj} \leq 2 \sum j = 0^{i-1} 2^{[bj]+1} \leq 2 \cdot 2^{[b^{i-1}]+2}$ where $[]$ denotes the integer party; this implies that $\frac{R_{i-1}}{r_i} \leq 1$ and goes to zero at infinity.

Since the function

$$f(x) = \frac{e^{(m-1)b_0} (1+x)^m}{e^{(m-1)b_0} (1+x)^m - 1}$$

is nonincreasing on $[0, 1]$ we have:

$$\frac{2^m e^{(m-1)b_0}}{2^m e^{(m-1)b_0} - 1} \leq f(x) \leq \frac{e^{(m-1)b_0}}{e^{(m-1)b_0} - 1}$$

hence

$$\theta_i \geq \frac{2^m e^{(m-1)b_0}}{2^m e^{(m-1)b_0} - 1} \theta_{i-1}.$$

which means :

$$\theta_i \geq a^i \theta_0$$

where

$$a = \frac{2^m e^{(m-1)b_0}}{2^m e^{(m-1)b_0} - 1}$$

and $\theta_0 = \text{vol}B(p, 2)$

Let R be a sufficient large positive number; there exists i such that $r_{i-1} \leq R \leq r_i$, otherwise

$$r_i = 2^{b^i} \Rightarrow i = \frac{\ln\left(\frac{\ln r_i}{\ln 2}\right)}{\ln b} \geq \frac{\ln(\ln R)}{\ln b}$$

$$\theta_i \geq C.a^{\frac{\ln\left(\frac{\ln r_i}{\ln 2}\right)}{\ln b}} \geq C.a^{\frac{\ln(\ln R)}{\ln b}} = C(\ln(R))^\rho$$

Where C, C_1 are positive constants and $\rho = \frac{\ln a}{\ln b} > 0$

Those results give a hope for showing the following conjecture due to S.H.Zhu and which is a version of Grove and Peterson conjecture's:

Conjecture: Given $c > 0$, do there exist constants $\epsilon(m, c)$ and $R(m, c)$ such that if

$$K(x) \geq -\lambda(d(p, x)), \text{Vol}(B(p, r)) \geq cr^m,$$

and

$$\int_0^\infty t\lambda(t)dt \leq \epsilon,$$

then any metric ball of radius r is contractible in the cocentric ball of radius $R.r$? Is M diffeomorphic to \mathbf{R}^m ?

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